Preliminary:

$\mathbb{Q}$ = Set of rationals

$= \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \right\}$, $\mathbb{Z}$ - set of integers. There are numbers other than rationals.

Consider $(p/q)^2 = 2$, $(p, q) = 1$.

$$p^2 = 2q^2 \Rightarrow p = 2m \Rightarrow 2m^2 = q^2$$
$$\Rightarrow q = 2n \Rightarrow (p, q) \geq 2,$$ which is a contradiction. Thus, $\cancel{\frac{p}{q}} = \sqrt{2}$ is not a rational number. Such no.s we say irrational & we denote $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ as the set of irrationals.

* Set of rationals is not complete in the following sense.

Def^n: Let $A \subset \mathbb{R}$. A number x_0 is called an upper bound for A if $a \leq x_0$, $\forall a \in A$. Similarly, y is called a lower bound for A if $\exists$ $y_0 \in \mathbb{R}$ s.t. $a \geq y_0$, $\forall a \in A$.

Def^n: An upper bound x_0 of A is called a least upper bound (l.u.b) or supremum if

z is any upper bound of A, implies $x_0 \leq z$. Similarly, greatest lower bound (or infimum) is defined. (2)

Notice that inf (or sup) need not required to belong to the set.

Ex. $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. $\inf A = 0$, $\sup A = 1$.

* Every non-empty subset of $\mathbb{R}$ having an upper bound has l.u.b (sup) and every non-empty set in $\mathbb{R}$ having a lower bound has glb (inf).
 This is known as Completeness property of $\mathbb{R}$. (For a proof, see Chap 1, Rudin)

Ex. If $A(\neq \phi) \subset \mathbb{R}$ is not bounded above, we write $\sup A = \infty$. Similarly, if $B \neq \phi$ is not bounded below, we write $\inf B = -\infty$.

If $A = \Phi$ (empty), then we write
 $\inf A = \infty$, and $\sup A = -\infty$.

(Hint: $\{a\} \subset \{a, b\} \Rightarrow \inf\{a\} = a \geq \inf\{a, b\}$)
 $\therefore \phi \subset \{a\} \Rightarrow \inf \phi \geq a, \forall a \in \mathbb{R}$)

Ex. $A \subset B \subset \mathbb{R} \Rightarrow \inf A \geq \inf B$ & $\sup A \leq \sup B$.

Archimedean property:

let $x > 0$ & y be any real no. Then $\exists$ a positive integer n st $nx > y$.

~~proof~~ (i.e any two real numbers can be compared).

proof: If $\nexists$ any $n \in \mathbb{N}$ s.t. $nx > y$. Then Suppose $nx \leq y$, $\forall\, n \in \mathbb{N}$. Thus, y is an upper bound of the set $\{nx : n \in \mathbb{N}\}$. By completeness of $\mathbb{R}$ $\exists\, \alpha \in \mathbb{R}$ st. $\alpha = \sup\{nx : n \in \mathbb{N}\} \leq y$.

$\exists\, n \in \mathbb{N}$ st $\alpha - x < nx < \alpha \Rightarrow \alpha < (n+1)x$, which contradicts the fact that α is a supremum.

Ex. $A = \{r \in \mathbb{Q} : r > 0$ & $r^2 < 2\}$. Then $\sup A = \sqrt{2} \notin \mathbb{Q}$.

* If $x, y \in \mathbb{R}$, then $x < y$ (or $x > y$).
$y - x > 0$. Comparing $y - x$ with 1 (apply Archimedean principle or AP), we get
$n(y-x) > 1$.

$\Rightarrow \exists$ integer m st $ny > m > nx$

$\Rightarrow x < \frac{m}{n} < y.$

ie between any two real there is a rational. Similarly, $\frac{x}{\sqrt{2}} < \frac{m}{n} < \frac{y}{\sqrt{2}}$

$\Rightarrow$ between any two real, there is an irrational.

Ex. Find Inf & Sup of $\left\{ \frac{m}{m+n} : m, n \in N \right\}$.

Let $A = \left\{ \frac{m}{m+n} : m, n \in N \right\}$. Clearly,

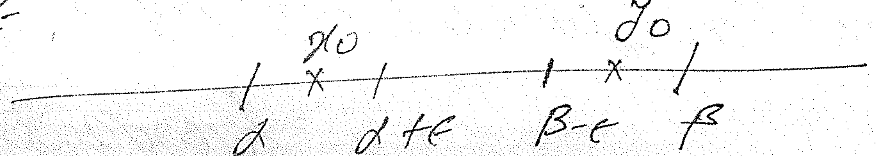
$\left\{ \frac{1}{1+n} : n \in N \right\} \subset A$ & $\frac{1}{1+n}$ approaches to zero for large n. If $\alpha = \text{Inf} A > 0$. Then by AP, $\exists\ n \in N$ st $(n+1)\alpha > 1$.

$\Rightarrow \alpha > \frac{1}{1+n}$ ✗.

If $\beta = \sup A < 1$. Then $(m+1)(1-\beta) > 1$ (by AP).

$\Rightarrow \beta < \frac{m}{m+1}$ ✗.

Ex. If $\alpha = \text{Inf} A$ & $\text{Sup} A = \beta$. Then for each $\epsilon > 0$, $\exists\ x_0, y_0 \in A$ st $x_0 < \alpha + \epsilon$ & $y_0 > \beta - \epsilon$.



Proof: Suppose for a given $\epsilon > 0$, $\nexists$ $a \in A$ s.t. $a < \alpha + \epsilon$. Then $a \geq \alpha + \epsilon$, $\forall a \in A$.

$\Rightarrow a > \alpha + \epsilon > \alpha \Rightarrow \alpha + \epsilon$ is a lower bound, which contradict the fact that α is the greatest lower bound. Similarly for β.

Def^n: A function $f: \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called sequence, and we write

$$\{f(1), f(2), \ldots, f(n), \ldots\} \text{ or } \{f_n\}.$$

A sequence $(a_n) \subset \mathbb{R}$ is said to be convergent to l if $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t.

$$n \geq n_0 \Rightarrow |a_n - l| < \epsilon$$

$$\text{or } a_n \in (l-\epsilon, l+\epsilon).$$

Ex. $a_n = \frac{1}{n} \to 0$. For this let $\epsilon > 0$,

$$\frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon} \Rightarrow \left[\frac{1}{\epsilon}\right].$$

$$\Rightarrow \forall n > \left[\frac{1}{\epsilon}\right] + 1 = n_0.$$

$$|a_n - 0| < \epsilon.$$

Result: If x_n is $\uparrow$ & bounded above, then x_n is conv^t and $\lim x_n = \sup x_n$.

proof: Let $\alpha = \sup x_n$. Then for $\epsilon > 0$,

$\exists\, x_{n_0}$ s.t. $x_{n_0} > \alpha - \epsilon$.

$$\Rightarrow \alpha + \epsilon > x_n \geq x_{n_0} > \alpha - \epsilon, \quad \forall n \geq n_0.$$

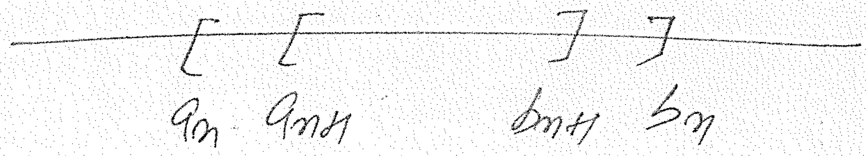
Thus, $x_n \to \alpha$.

Similarly, if $x_n \downarrow$ & bounded below, then x_n is conv & $\lim x_n = \inf x_n$.

Nested Interval theorem:

If $I_n \supset I_{n+1} \supset \cdots$ & $\lim (b_n - a_n) = 0$, where $I_n = [a_n, b_n]$. Then $\bigcap I_n = \{x\}$.

Proof:



$\Rightarrow a_n \uparrow$ & $< b_1$ and $b_n \downarrow\ > a_1$.

Hence, $\{a_n\}$ & $\{b_n\}$ are conv & let $a_n \to a$ & $b_n \to b$. Then

$$b - a = \lim (b_n - a_n) = 0 \Rightarrow a = b.$$

Notice that $a_n \leq a$ & $b_n \leq a$

$$\Rightarrow a_n \leq a \leq b_n \Rightarrow a \in \bigcap I_n.$$

If $x \in \bigcap I_n$, then $a_n \leq x \leq b_n \Rightarrow x = a$.

Df. $\{x_n\}$ is a sequence & $n_1 < n_2 < \cdots$ where $n_k \in N$. Then $\{x_{n_k}\}$ is called a subsequence of seq^n $\{x_n\}$.

Ex. $\{\frac{1}{k^2}\}$, $\{\frac{1}{2^k}\}$ are subsequences of $\{\frac{1}{n}\}$, with $n_k = k^2$, & $n_k = 2^k$ resp.

Bolzano-Weierstrass theorem:

Every bounded sequence in $\mathbb{R}$ has a conv. subsequence.

Proof: Let $x_n \in [a,b]$ ($\because$ x_n is bdd).
Divide $[a,b]$ into two parts, say $[a,b_1]$ & $[b_1,b]$, and write

$$[a,b] = I_1 \cup I_1'.$$

Suppose I_1 contains ∞ly many terms of (x_n). Choose $x_{n_1} \in I_1$. Further, divide $I_1 = I_2 \cup I_2'$ & suppose I_2 contains ∞ly many terms of (x_n). Choose $x_{n_2} \in I_2$.
Then $x_{n_k} \in I_k$ & $I_k \supset I_{k+1} \supset \cdots$
By NIT, $\bigcap I_k = \{x\}$. Thus, for each $\epsilon > 0$, $\exists$ $k_0 \in N$ s.t. $\forall$ $k > k_0 \Rightarrow I_k \subset (x-\epsilon, x+\epsilon)$

That is, $x_{n_K} \in (x-\epsilon, x+\epsilon)$, $\forall\ K \geq K_0$. (8)

$\Rightarrow x_{n_K} \to x$.

<u>Alternative proof:</u>

<u>We have</u> $(x_n) \subset [a,b]$.

Let $x_{n_K} = \inf\limits_{n \geq K} x_n = \inf\{x_K, x_{K+1}, \dots\}$.

Then $x_{n_K} \uparrow \ \& < b. \Rightarrow x_{n_K} \to \sup\limits_{K \geq 1}\left(\inf\limits_{n \geq K} x_n\right)$.

i.e. $x_{n_K} \to \underline{\lim}\, x_n$. Similarly, write

$y_{n_K} = \sup\limits_{n \geq K} x_n = \sup\{x_K, x_{K+1}, \dots\}$

Then. $y_{n_K} \downarrow \ \& > a \Rightarrow y_{n_K} \to \inf\limits_{K \geq 1}\left(\sup\limits_{n \geq K} x_n\right)$.

i.e. $y_{n_K} \to \overline{\lim}\, x_n$.

Exercise: (i) Show that $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$.

(ii) If $x_n \to x$, then $\underline{\lim}\, x_n \geq \overline{\lim}\, x_n$.

(i.e $\underline{\lim}\, x_n = x = \overline{\lim}\, x_n$).

Example. $x_n = (-1)^n$. $\underline{\lim}\, x_n = -1 < 1 = \overline{\lim}\, x_n$.

Thus, deduce that x_n is conv iff $\underline{\lim}\, x_n = \overline{\lim}\, x_n$.

Ex. If $X_n = (x_n, y_n) \in \mathbb{R}^2$ is bdd seq^n, then

$\sqrt{x_n^2 + y_n^2} \leq M, \ \forall n \geq 1.$

$\Rightarrow |x_n| \leq M$ & $|y_n| \leq M$. By B-W, $\exists\, x_{n_k}$ s.t. $x_{n_k} \to x \in \mathbb{R}$. Now, (x_{n_k}, y_{n_k}) is also a bounded seq^n in $\mathbb{R}^2$. Thus, y_{n_k} is bounded & by B-W, $y_{n_{k,l}} \to y \in \mathbb{R}$. Thus,

$(x_{n_{k,l}}, y_{n_{k,l}}) \to (x, y)$.

Def^n: A set $A \subset \mathbb{R}$ is said to be open if $\forall x \in A$ encloses an ~~int~~ open interval $I_x \subset O$.

Thus, the countable union of open intervals is open. On the other hand any open set in $\mathbb{R}$ can be written as countable union of open intervals.

Theorem: Every open set in $\mathbb{R}$ can be uniquely expressed as the countable union of disjoint open intervals.

Proof: Let O be an open subset of $\mathbb{R}$.

For $x \in O$, let $a_x = \inf\{a : (a, x] \subset O\}$.

and $b_x = \sup\{b : [x, b) \subset O\}$. Then,

$I_x = (a_x, b_x)$ will be the largest open interval containing x. Now, if $x, y \in O$ & $x \neq y$. Then either $I_x \cap I_y = \Phi$ or $I_x = I_y$. (10)

If $I_x \cap I_y \neq \Phi$. Then $I_x \cup I_y$ is an open interval containing x & y. Therefore, by maximality of I_x & I_y, it follows that

$$I_x \cup I_y \subseteq I_x \quad \& \quad I_x \cup I_y \subseteq I_y$$

$$\Rightarrow I_y \subseteq I_x \quad \& \quad I_y \subseteq I_x.$$

$$\Rightarrow I_x = I_y.$$

Now, $O = \bigcup_{x \in O} I_x$. Since I_x & I_y are disjoint (if $x \neq y$), we can assign distinct rational to each of them. Thus,

$\{I_x : x \in O\} \longleftrightarrow \{r_x \in Q : r_x \in O\} \subseteq Q$.

Hence, $O = \bigcup_{i=1}^{\infty} I_{r_i}$. —(1)

The representation (1) is unique.

Let $O = \bigcup_{n=1}^{\infty} I_n = \bigcup_{m=1}^{\infty} J_m$.

Then $I_n = I_n \cap O = \bigcup_{m=1}^{\infty} (I_n \cap J_m)$.

Since $\{I_n \cap I_m : m \in N\}$ is a disjoint collection of open interval, it follows that $I_n \subset I_n \cap I_{m_0}$, for some m_0. Similarly, $I_{m_0} \subset I_{n'} \cap I_{m_0}$ for some n'. Thus, $I_n \subset I_{m_0} \subset I_{n'}$. But $I_n \cap I_{n'} \neq \Phi$ (by maximality of I_n's), it follows that $I_n = I_{m_0} = I_{n'}$.

<u>Closed Set</u>:

A set $A \subseteq \mathbb{R}$ is said to be closed if for each ~~x∈A, the~~ sequence $x_n \in A$ s.t $x_n \to x$, implies $x \in A$.

Ex. A set $F \subset \mathbb{R}$ is closed iff F^c is open.

proof: Let F be closed. Suppose F^c is not open. Then ~~∄ ϵ>0 s.t (x-~~ for some $x \in F^c$, $\nexists\ \epsilon > 0$ s.t $(x-\epsilon, x+\epsilon) \subset F^c$.

Take $\epsilon = \frac{1}{n}$, then $x_n \in (x-\frac{1}{n}, x+\frac{1}{n})$ & $x_n \in F$. Thus, $x_n \to x$ & F is closed, implies $x \in F$, which is a contradiction. Hence, F^c is open.

Conversely, suppose F^c is open. Let $x_n \in F$, & $x_n \to x$. Claim $x \in F$.

If $x \notin F$, then $x \in F^c$, which is open. Then $\exists\ r > 0$ s.t. $(x-r, x+r) \subset F^c$. Since $x_n \to x$, $\exists\ n_0 \in \mathbb{N}$ s.t.

$$n \geq n_0 \Rightarrow x_n \in (x-r, x+r) \subset F^c,$$

which is absurd. Thus, $x \in F$.

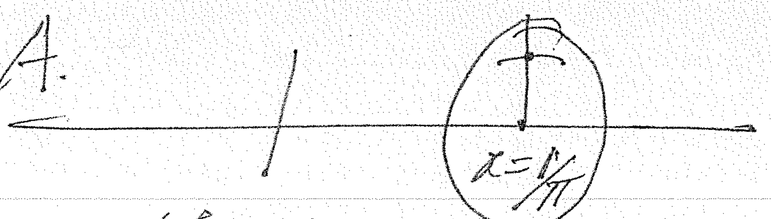
Notice that we can define open sets & closed sets in $\mathbb{R}^n$.

Ex. The set $A = \{(x,y) : y = \sin\frac{1}{x}, x \neq 0\}$ is neither open nor closed in $\mathbb{R}^2$.

Let $x_n = \frac{1}{n\pi}$; $n \in \mathbb{N}$, then $(x_n, y_n) = (\frac{1}{n\pi}, 0) \in A$.

But $\lim (x_n, y_n) = (0,0) \notin A$.

$B_{1/n}(\frac{1}{\pi}, 0) \not\subset A$.



($\because$ A is the graph of a function $f(x) = \sin\frac{1}{x}$, $x \neq 0$)

<u>Interior of a set</u>:

Interior of a set A is the largest open set A° that contained in A. That is, if O is open & $O \subset A \Rightarrow O \subseteq A^\circ$.

Ex. $\mathbb{N}^{\circ} = \Phi$, $\mathbb{Q}^{\circ} = \Phi$, $(\mathbb{R}\setminus\mathbb{Q})^{\circ} = \Phi$.

and $\{(x,y) : y = \sin\frac{1}{x}, x \neq 0\}^{\circ} = \Phi$.

Closure of a set:

Closure of a set A is the smallest closed set $\bar{A}$ that contains A. That is, if B is closed & $A \subseteq B \Rightarrow \bar{A} \subseteq B$.

~~Excercise~~ Exercise: Show that:

$\bar{A} = \overline{\{(x,y) : y = \sin\frac{1}{x}, x \neq 0\}} = \{(x,y) : y = \sin\frac{1}{x}, x \neq 0\}$ $\cup (\{0\} \times [-1,1])$ (Hint: Away from $(0,0)$, A is a graph of continuous function). Also, RHS is a closed set containing A. Thus, $\bar{A} = $ RHS. (by def^n).

Def^n: A closed & bounded subset $K \subset \mathbb{R}^n$ is called a compact subset of $\mathbb{R}^n$.

Ex. The set $\{(x,y) : y = \sin\frac{1}{x}, x \neq 0\} \cup (\{0\} \times [-1,1])$ is closed but not bounded ~~as it contains $\mathbb{R} \times \{A\}$, when A is a non empty subset of $[-1,1]$~~, every $x \in \mathbb{R}$, $\exists y \in [-1,1]$. (& replace $\sin\frac{1}{x}$ by $\frac{1}{x}$)

However, using B-W, it can be deduced that a set $K \subset \mathbb{R}^n$ is compact iff every open cover of K reduces to ~~a~~ finite sub-cover.

i.e. if $K \subset \bigcup_{i \in I} O_i$, then $K \subset \bigcup_{k=1}^{n} O_{i_k}$.

Ex. A set $F \subset \mathbb{R}$ is closed iff $\forall \epsilon > 0$,

$(x-\epsilon, x+\epsilon) \cap F \neq \emptyset \Rightarrow x \in F$.

Def^n: A set $A \subset \mathbb{R}$ is said to be dense if $\forall x \in \mathbb{R}$, $\exists x_n \in A$ s.t. $x_n \to x$. (i.e. every pt of $\mathbb{R}$ is the limit of sequence from A).

In this case, we write $\overline{A} = \mathbb{R}$.

Ex. $x \in \mathbb{R}$, then

$$x = x_0 + \frac{x_1}{10} + \cdots + \frac{x_n}{10^n} + \cdots \infty$$

~~Let x~~ where $x_i \in \{0, 1, 2, \ldots, 9\}$.

Let $S_n = x_1 + \cdots + \frac{x_n}{10^n}$. Then $S_n \in \mathbb{Q}$, & $S_n \to x$. Thus $\overline{\mathbb{Q}} = \mathbb{R}$.

Ex. Show that $\left\{ \frac{k}{2^n} : k = 0, 1, 2, \ldots 2^n, n \in \mathbb{N} \right\}$ is dense in $[0, 1]$.

Drawback of Riemann integration

Let $f:[a,b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}[a,b]$ (i.e f is Riemann integrable) iff f is almost continuous.

However, there are functions which neither almost cont, nor bounded etc.

(I) $f:[0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 1 & x \in (\mathbb{R}\setminus\mathbb{Q}) \cap [0,1] \\ 0 & x \in \mathbb{Q} \cap [0,1] \end{cases}$

Then $\inf_P U(P,f) = 1$ & $\sup_P L(P,f) = 0$.

$\Rightarrow f \notin \mathcal{R}[0,1]$. Hence, there is scope to consider one of them and avoid other.

(II) $\int_0^1 \frac{1}{\sqrt{t}}\,dt$, $f(t) = \frac{1}{\sqrt{t}}$ is not bounded near "0"! However, $\int_{1/n}^1 \frac{1}{\sqrt{t}}\,dt = 2(1 - \frac{1}{\sqrt{n}}) \le 2$.

Question: Should we write $\int_0^1 \frac{1}{\sqrt{t}}\,dt = \sup_n \int_{1/n}^1 \frac{1}{\sqrt{t}}\,dt = 2$.

(III) $\int_0^\infty \frac{1}{1+t^2}\,dt$, $\int_0^n \frac{1}{1+t^2}\,dt = \tan^{-1} n \le \pi/2$.

Does it suitable to write

$$\int_0^\infty \frac{1}{1+t^2}\,dt = \sup_n \int_0^n \frac{1}{1+t^2}\,dt = \frac{\pi}{2} \, ?$$

Remark: If we ~~only~~ consider only $U(P,f)$, we are overestimating the area of curve by rectangular cover. Hence, for the real line case, we think of over-estimating the length by intervals (covering the set by intervals).

(16)

Lebesgue outer measure:

For open interval $I = (a,b)$, we assign the length $l(I) = b-a$. For $I = (a,\infty)$ or $(-\infty, b)$, assign $l(I) = \infty$.

Now, the question is to assign an appropriate length to an arbitrary subset of $\mathbb{R}$. If $O \subset \mathbb{R}$ is open, then $O = \dot{\bigcup}_{n=1}^{\infty} I_n$, $I_n = (a_n, b_n)$, & $I_n \cap I_m = \phi$ if $n \neq m$. In this case, we can consider $l(O) = \sum_{n=1}^{\infty} l(I_n)$. However, if $A \subseteq \mathbb{R}$, $A \subseteq O \subseteq \mathbb{R}$. Hence, $A \subset \dot{\bigcup}_{n=1}^{\infty} I_n$. Thus, we have an over-estimate ~~of~~ for length of A.

i.e. $l(A) \leq \sum l(I_n)$, s.t. $A \subset \dot{\bigcup}_{n=1}^{\infty} I_n$.

$\Rightarrow l(A) \leq \inf\{\sum l(I_n) : A \subset \cup I_n\}$.

Before, we come to the formal definition of Lebesgue outer measure, we can

See relation between disjoint & overlapping intervals.

Lemma: Let $\{I_n\}$ be a ~~disjoint~~ family of disjoint open intervals, & $\{J_m\}$ be any collection of countably many open intervals, such that $\dot{\bigcup}_{n=1}^{\infty} I_n = \bigcup_{m=1}^{\infty} J_m$. Then

$$\sum_{n=1}^{\infty} l(I_n) \leq \sum_{m=1}^{\infty} l(J_m). \quad \text{———} (*)$$

Proof: On contrary, let $\sum_{n=1}^{\infty} l(I_n) > \sum_{m=1}^{\infty} l(J_m)$.

Then $\exists$ $N \in \mathbb{N}$ s.t.

$$\sum_{n=1}^{N} l(I_n) > \sum_{k=1}^{\infty} l(J_k) \quad \text{———} (1)$$

(if $\nexists$ $N \in \mathbb{N}$ s.t. (1) holds, then (*) is true, as left side of (1) is an ↑ seq. (in N etc.)

Then $\exists$ $\epsilon, \epsilon' > 0$ s.t.

$$\sum_{n=1}^{N} l(I_n) - \sum_{n=1}^{N} \frac{2\epsilon}{2^n} > \sum_{k=1}^{\infty} l(J_k) + \sum_{k=1}^{\infty} \frac{2\epsilon'}{2^k}$$

$$\Rightarrow \sum_{n=1}^{N} l(I_n') > \sum_{k=1}^{\infty} l(J_k'), \text{ where}$$

$I_n' = [a_n + \frac{\epsilon}{2^n}, b_n - \frac{\epsilon}{2^n}]$, $I_n = (a_n, b_n)$,

& $J_k' = (c_k - \frac{\epsilon}{2^k}, d_k + \frac{\epsilon'}{2^k})$, $J_k = (c_k, d_k)$.

$\Rightarrow \dot{\bigcup}_{n=1}^{N} I_n' \subset \bigcup_{k=1}^{\infty} J_k'$. Since LHS is compact, $\exists$ $M \in \mathbb{N}$ s.t. $\dot{\bigcup}_{n=1}^{N} I_n' \subset \bigcup_{k=1}^{M} J_k'$.

By induction, it can be shown that

$$\sum_{n=1}^{N} l(I_n') \leq \sum_{k=1}^{M} l(J_k') < \sum_{k=1}^{\infty} l(J_k'). \qquad (18)$$

$$\Rightarrow \sum_{n=1}^{N} l(I_n) < \sum_{k=1}^{\infty} l(J_k) + \sum_{n=1}^{N} \frac{2\epsilon}{2^n} + \sum_{k=1}^{\infty} \frac{2\epsilon'}{2^k}$$

But this is true for every small choice of ϵ & $\epsilon' > 0$.

Thus, $\sum_{n=1}^{N} l(I_n) \leq \sum_{k=1}^{\infty} l(J_k)$. This contradicts (1).

Hence, $\sum_{n=1}^{\infty} l(I_n) \leq \sum_{k=1}^{\infty} l(J_k)$.

Now, it is appropriate to define

$m^*(A) = \inf\left\{\sum_{n=1}^{\infty} l(I_n) : A \subset \bigcup I_n\right\}$, where

I_n's are open intervals, need not be disjoint.

Moreover, I_n could be any type of intervals (e.g. $I_n = [a_n, b_n]$, $(a_n, b_n]$, $[a_n, b_n)$).

Since, empty set $\Phi \subset (0, \epsilon)$, $\forall \epsilon > 0$,

$m^*(\Phi) \leq \epsilon$, $\forall \epsilon > 0 \Rightarrow m^*(\Phi) = 0$.

For $\{a\} \subset R$, $\{a\} \subset (a - \epsilon/2, a + \epsilon/2)$

$\Rightarrow m^*(\{a\}) \leq \epsilon$, $\forall \epsilon > 0$

$\Rightarrow m^*(\{a\}) = 0$.