Absolute Continuity of the integral:

If $f \in L^1(X, S, \mu)$, we have seen that $\nu(E) = \int_E f \, d\mu$ defines a set function which is countably additive on (X, S).

Moreover, $|\nu(E)| \leq \int_E |f| \, d\mu < \infty$ ($\because f \in L^1$).

This shows that $|\nu(E)|$ is small if $\mu(E)$ is small. Thus, we can prove the following result.

Theorem: Let $f \in L^1(X, S, \mu)$. Then for $\epsilon > 0$, $\exists$ a $\delta > 0$ & set $E \in S$ such that

$$\mu(E) < \delta \Rightarrow \int_E |f| \, d\mu < \epsilon \quad (\& \ |\nu(E)| < \epsilon).$$

Proof: For $n \in \mathbb{N}$, we have

$$\int_X |f| \, d\mu = \int_{\{x : |f(x)| \leq n\}} |f| \, d\mu + \int_{\{x : |f(x)| > n\}} |f| \, d\mu < \infty.$$

We all need to prove that 2nd integral in R.H.S. is small, while n is large.

Let $E_n = \{x \in X : |f(x)| > n\}$. Then $E_n \downarrow E$, where $E = \{x \in X : |f(x)| = \infty\}$. Since $f \in L^1$,

$$\lim \mu(E_n) = \mu(E) = 0. \quad \text{Next,}$$

$$\lim_{n \to \infty} (\chi_{E_n}(x) - \chi_E(x)) = \lim_{n \to \infty} \chi_{E_n \setminus E}(x) = 0$$

(Hint: if $x \notin E$, then $\exists \ n_0 \in \mathbb{N}$ s.t. $x \notin E_n, \forall n \geq n_0$

Hence, $f\chi_{E_n} \to f\chi_E = 0$ a.e on X.
Since $|f\chi_{E_n}| \le |f| \in L^1$, by DCT, it follows that $\lim\limits_{n\to\infty}\int_X |f\chi_{E_n}| = 0$. That is,

$$\lim \mu(E_n) = 0 \Rightarrow \lim \int_{E_n} |f|\,d\mu = 0.$$ Hence,

$\forall\, \epsilon > 0$, $\exists\, \delta > 0$ & $n_0 \in \mathbb{N}$ s.t

$n \ge n_0$, $\mu(E_n) < \delta \Rightarrow \int_{E_n} |f|\,d\mu < \epsilon$. In particular,

$$\mu(E_{n_0}) < \delta \Rightarrow \int_{E_{n_0}} |f|\,d\mu < \epsilon.$$

<u>Bounded Convergence theorem (BCT):</u>

Let (X, S, μ) be a finite measure space. If f_n & f are mble functions on (X, S, μ) s.t

(i) $|f_n(x)| \le M$, $\forall\, n \ge 1$, $\forall\, x \in X$, and

(ii) $f_n \to f$ point-wise on X.

Then $\int_X f\,d\mu = \lim\limits_{n\to\infty}\int_X f_n\,d\mu$.

<u>Proof:</u> Since $\mu(X) < \infty$, $\int_X |f_n| \le M\mu(X) < \infty$.
Hence f_n is dominated by $M \in L^1(X)$. By DCT, $\int_X f\,d\mu = \lim \int_X f_n\,d\mu$.

Now, with the help of MCT, Fatou's lemma, DCT & BCT, we shall compare Riemann integral with Lebesgue integral.

Comparison of Riemann integral with Lebesgue integral:

Let $f:[a,b] \longrightarrow \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and let $\Delta x_i = x_{i-1} - x_i$, $M = \sup\limits_{x_{i-1} < x \leq x_i} f(x)$ and

$m_i = \inf\limits_{x_{i-1} < x \leq x_i} f(x)$. Write $U(P,f) = \sum\limits_{i=1}^{n} M_i \Delta x_i$

and $L(P,f) = \sum\limits_{i=1}^{n} m_i \Delta x_i$. Since f is bounded, $\exists\ m, M > 0$ s.t. $m \leq f(x) \leq M$, $\forall x \in [a,b]$.

Hence

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a) \quad \text{---} (*)$$

The function f is said to be Riemann integrable (or $f \in R[a,b]$) if

$$\inf_P U(P,f) = \sup_P L(P,f)$$

$$\Rightarrow \inf_P w(P,f) = \inf_P \{U(P,f) - L(P,f)\} = 0.$$

Hence, for each $\epsilon > 0$, $\exists$ a partition P s.t. $w(P,f) < \epsilon$. Also for $\epsilon = 1/n$, $n \in \mathbb{N}$, $\exists$ a partition P_n s.t. $w(P_n, f) < \frac{1}{n}$, $\forall n \in \mathbb{N}$.

Hence $\lim\limits_{n\to\infty} w(P_n, f) = 0$.

Theorem: Let $f:[a,b] \longrightarrow \mathbb{R}$ be a bounded function. Then $f \in R[a,b]$ iff $\exists$ a seq of partitions P_n such that $\lim\limits_{n\to\infty} w(P_n, f) = 0$.

Proof: We have already seen the forward implication. For other one, if $\lim w(P_n,f)=0$, then for each $\epsilon>0$, $\exists\ n_0\in\mathbb{N}$ s.t. 125

$\forall\ n\geq n_0 \Rightarrow w(P_n,f)<\epsilon$. But then

$$\inf_P w(P,f)\leq w(P_{n_0},f)<\epsilon,\ \forall\ \epsilon>0.$$

Hence, $\inf_P w(P,f)=\inf_P\{U(P,f)-L(P,f)\}=0$.

Since f is bounded, $\inf_P U(P,f)=\sup_P L(P,f)$.

Ex. Let $f:[0,1]\longrightarrow\mathbb{R}$, $f(x)=\begin{cases}1 & x=1/2\\ 0 & x\neq 1/2.\end{cases}$

Then f is bounded and for $P_n=\{\frac{i}{n}: i=0,1,2,\dots,n\}$

$$w(P_n,f)=\sum_{i=1}^{n}(M_i-m_i)\Delta x_i\leq 2\max_{a<i\leq n}\Delta x_i<2\cdot\frac{1}{n}\to 0.$$

(Hint: $1/2\in[x_{i-1},x_i]\cup[x_i,x_{i+1}]$).

If $f:[a,b]\longrightarrow\mathbb{R}$ is continuous. Then $f\in R[a,b]$.

Since f is cont, f is unif cont on $[a,b]$. For each $\epsilon>0$, $\exists\ \delta>0$ s.t.

$$|s-t|<\delta\Rightarrow|f(s)-f(t)|<\epsilon.$$

Choose a partition P s.t. $\Delta x_i<\delta$. Then

$$-\epsilon<f(t)-f(s)<\epsilon,\quad\forall\ s,t\in[x_{i-1},x_i].$$

By taking sup and then inf, we get

$$-\epsilon\leq M_i-m_i\leq\epsilon,\ \forall\ i=1,2,\dots,n.$$

Hence, $w(P,f)=\sum_{i=1}^{n}(M_i-m_i)\Delta x_i\leq\sum_{i=1}^{n}\epsilon\Delta x_i=\epsilon(b-a)$.

Notice that if $P_1 \subset P_2$, then $L(P_1,f) \leq L(P_2,f)$ and $U(P_1,f) \geq U(P_2,f)$. Hence

$$w(P_1,f) \geq w(P_2,f).$$

Using this fact, it is enough to look at $\lim w(P_n,f) = 0$, while P_n is an $\uparrow$ seq^n.

<u>Theorem</u>: Let $f:[a,b] \longrightarrow \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}[a,b]$ iff $\exists$ an $\uparrow$ seq^n of partitions P_n of $[a,b]$ such that

$$\lim_{n\to\infty} w(P_n,f) = 0.$$

<u>Proof</u>: Since, $f \in \mathcal{R}[a,b]$, by previous theorem, $\exists$ a partitions P_n such that $\lim w(P_n,f) = 0$.

Now, let $Q_1 = P_1$, $Q_n = P_1 \cup P_2 \cup \cdots \cup P_n$. Then

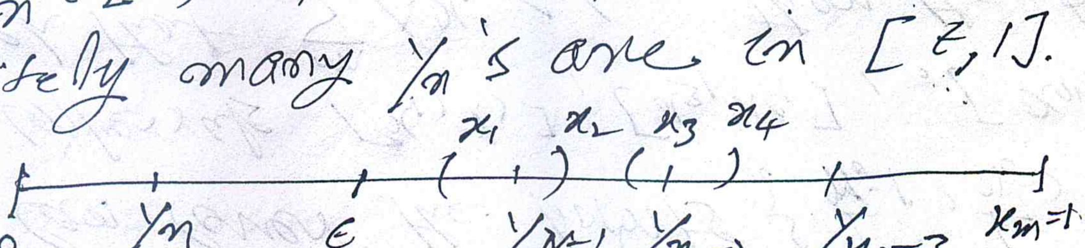
$$w(Q_n,f) \leq w(P_n,f) \longrightarrow 0.$$ Converse is obvious from previous theorem.

Ex. Let $f:[0,1] \longrightarrow \mathbb{R}$ be such that

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = 1/n \\ 0 & \text{o.w.} \end{cases}$$ Then $f \in \mathcal{R}[a,b]$

and $\int_0^1 f(x)\,dx = 0$.

Let $\epsilon > 0$. Since $\frac{1}{n} \to 0$, $\exists$ $n_0 \in \mathbb{N}$ such that $\frac{1}{n} \in [0,\epsilon]$, $\forall$ $n \geq n_0$. Hence, only finitely many $1/n$'s are in $[\epsilon,1]$.



We can cover the points $\{\frac{1}{n_0-1}, \frac{1}{n_0-2}, \dots 1\}$ by intervals $[x_1, x_2], [x_3, x_4], \dots, [x_{m-1}, x_m]$ such that $\sum_{i=2}^{m} \Delta x_i < \epsilon$. Then the partion $P = \{0, \epsilon, x_1, x_2, \dots, x_m = 1\}$ is desire, and

$$w(P, f) = \sum_{i=0}^{m} (M_i - m_i) \Delta x_i$$ (127)

$$= M_0 \Delta x_0 + \sum_{i=1}^{m} M_i \Delta x_i$$

$$< 1 \cdot \epsilon + \epsilon = 2\epsilon.$$

Hence, $f \in R[a, b]$ & $\int_0^1 f(x)dx = \sup_P L(P, f) = 0$.

<u>Theorem</u>: Let $f \in R[a, b]$. Then $f \in L^1[a, b]$ and $\underset{(L\text{-Integral})}{\int_{[a,b]} f(x) dm(x)} = \underset{(R\text{-Integral})}{\int_a^b f(x) dx}$.

Proof: Let $I = [a, b]$ and $f \in R(I)$. Then $\exists$ an $\uparrow$ seq^n of partions $P_n = \{x_0, x_1, \dots, x_{i-1}, x_i, x_{n_k}\}$ of I such that

$$\int_a^b f(x) dx = \lim U(P_n, f) = \lim L(P_n, f).$$

Let $\varphi_n = \sum_{i=1}^{n_k} M_i \chi_{(x_{i-1}, x_i]}$, where $M_i = \sup_{x_{i-1} < x \le x_i} f(x)$

and $\psi_n = \sum_{i=1}^{n_k} m_i \chi_{(x_{i-1}, x_i]}$, $m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$.

Then $\varphi_n \downarrow$ seq^n and $\psi_n \uparrow$ sequence.

Since, $f \in R(I)$, f is bounded. Hence

$\exists\, m, M > 0$ such that
$$m \le f(x) \le M, \quad \forall x \in I.$$

Then $\quad m \le \psi_n(x) \le f(x) \le \varphi_n(x) \le M, \quad$ —— (1)

for each $x \in I$. Note that for fixed $x \in I$, $\varphi_n(x) \downarrow$ seqn bounded below by m & $\psi_n(x) \uparrow$ seqn bounded above by M.

Hence, $\exists\, \varphi$ & ψ such that
$$\lim \varphi_n(x) = \varphi(x) \text{ \& } \lim \psi_n(x) = \psi(x).$$

Then φ & ψ (being limit of simple m'ble functions) are measurable, and
$$m \le \psi(x) \le f(x) \le \varphi(x) \le M, \quad \forall x \in I. \quad \text{—— (2)}$$

By BCT,
$$\int_I \varphi\, dm = \lim \int_I \varphi_n\, dm = \lim U(P_n, f) = \int_a^b f(x)\, dx$$

Similarly, $\int_I \psi\, dm = \int_a^b f(x)\, dx = \int_I \varphi\, dm$. Hence
$\int_I (\varphi - \psi)\, dm = 0$ iff $\varphi - \psi = 0$ a.e. ($\because \varphi - \psi \ge 0$)

That is, $\varphi = \psi$ a.e. & from (2), $f = \varphi = \psi$ a.e.

Hence, f is m'ble. Thus,
$$\int_I f\, dm = \int_I \varphi\, dm = \int_a^b f(x)\, dx.$$

<u>Remark</u>: If we assign a norm on $R[a,b]$ through $\|f\|_1 = \int_a^b |f(x)|\, dx$. Then by the previous result, $(R[a,b], \|\cdot\|_1) \subsetneq (L^1[a,b], \|\cdot\|_1)$

The conclusion is proper, because $f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

on $[0,1]$ is not R-integrable, because f is not bounded near "0". (Important!).

For $n \in \mathbb{N}$, if we write $f_n = f \chi_{[1/n, 1]}$. Then f_n increases to f pointwise. Hence, by

(*) MCT, $\int_{[0,1]} f \, dm = \lim \int_{[1/n, 1]} f \, dm = \lim \int_{1/n}^{1} f(x) dx = 2.$

Thus, $f \in L^1[0,1]$. In fact, the space $(R[0,1], \|\cdot\|_1)$ is an incomplete n.l.s, because for $m > n$, $\|f_n - f_m\|_1 = \int_{1/m}^{1/n} \frac{1}{\sqrt{x}} dx = 2\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right)$.
It implies $\{f_n\}$ is a Cauchy sequence in $R[0,1]$, but $\lim f_n(x) = f(x)$, $f \notin R[0,1]$. However, $\overline{R[0,1]} = L^1[0,1]$, that we see later.

Remark: Observation made in (*) is wider and can be generalized to the following result.

<u>Theorem</u>: Let $f : [a, \infty) \to \mathbb{R}$ be such that

(i) $f \in R[a,b]$, $\forall\ b > a$ & $b \in \mathbb{R}$,

(ii) $\int_{[a,b]} |f| \, dm \leq M < \infty$, $\forall\ b > a$

(M is independent of b).

Then $f \in L^1[a, \infty)$ &

$$\int_{[a,\infty)} f \, dm = \lim_{b \to \infty} \int_{[a,b]} f \, dm = \lim_{b \to \infty} \int_a^b f(x) dx.$$

Proof: Let $f_n = f\chi_{[a,n]}$. Then $f_n \to f$ pointwise on $I = [a,\infty)$. Notice that $\int_I |f_n|\,dm$ is an increasing sequence which is bounded above by M. Hence (130)

$$\lim \int_I |f_n|\,dm \leq M.$$

Since $|f_n| \uparrow |f|$ pointwise, by MCT,

$$\int_I |f|\,dm = \lim \int_I |f_n|\,dm \leq M < \infty.$$

Hence, $f \in L^1(I)$. Further, $|f_n| \leq |f| \in L^1(I)$, by DCT,
$$\int_I f\,dm = \lim \int_I f_n\,dm = \lim_{n\to\infty} \int_{[a,n]} f\,dm = \lim_{n\to\infty} \int_a^n f(x)\,dx.$$

Ex. Consider $f_n : \mathbb{R} \to \mathbb{R}$ by
(i) $f_n(t) = \frac{1}{n} e^{-t^2}$, (ii) $f_n(t) = \frac{1}{1+nt^2}$,
(iii) $f_n(t) = e^{-\frac{|t|}{n}} \chi_{(0,n)}(t)$, (iv) $f_n(t) = e^{-\frac{t^2}{n}}$.
If $f_n \to f$, check for $f \in L^1(\mathbb{R}, M, m)$, and commutation of limit & integral.

Characterization of R-integrable functions:

Theorem: Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then $f \in R[a,b]$ iff f is continuous on $[a,b]$ a.e. m.

Proof: Let $f \in R[a,b]$. Then $\exists$ an $\uparrow$ seq^n of partitions P_n of $[a,b]$ such that
$$\psi_n = \sum_{i=1}^{n_k} M_i \chi_{(x_{i-1}, x_i]} \downarrow f \;\&\; \varphi_n = \sum_{i=1}^{n_k} m_i \chi_{(x_{i-1}, x_i]} \uparrow f$$

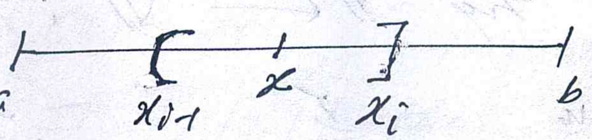
point wise almost everywhere on $[a,b]$. (By previous theorem). Consider

$\varphi_n \downarrow f$ & $\psi_n \uparrow f$ p.w. on $A^c \subset [a,b] = I$.

Then $m(A) = 0$. Note that partitioning pts of I could also be point of dis continuity. Hence, let $D = A \cup \left(\bigcup_{n=1}^{\infty} P_n\right)$. If $x \in I \setminus D$, then for each $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$\left.\begin{array}{l} \varphi_n(x) - f(x) < \epsilon \\ f(x) - \psi_n(x) < \epsilon \end{array}\right\} n \geq n_0$$

Since, φ_n & ψ_n are simple functions, $\exists$ $\{x_{i-1}, x_i\} \subset P_n$ for some $n \geq n_0$ such that $x \in [x_{i-1}, x_i]$ & $M_i - f(x) < \epsilon$ & $f(x) - m_i < \epsilon$.



For $y \in [x_{i-1}, x_i]$, we get

$$-\epsilon < m_i - f(x) \leq f(y) - f(x) \leq M_i - f(x) < \epsilon.$$

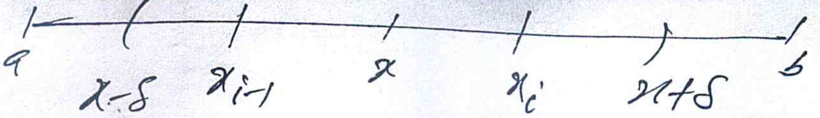
Hence f is cont at $x \in I \setminus D$, & $m(D) = 0$. Thus, f is continuous a.e. on I.

Conversely, suppose f is a.e. continuous on I. let $x \in (a,b)$ be a point of continuity of f. Then for each $\epsilon > 0$, $\exists$ $\delta > 0$ such that

(1) — $f(x) - \epsilon < f(y) < f(x) + \epsilon$ on $|x - y| < \delta$.

Suppose $\|P_n\| \to 0$, then for $\delta > 0$, $\exists$ $n_0 \in \mathbb{N}$

Such that $\|P_n\| < \delta$, $\forall\, n \geq n_0$.



From (I), $\quad f(x) - \epsilon \leq \inf\limits_{y \in [x_{i-1}, x_i]} f(x) \leq f(x) + \epsilon$

i.e. $\quad f(x) - \epsilon \leq m_i \leq f(x) + \epsilon$

Since $x \in (x_{i-1}, x_i]$, it follows that

$$f(x) - \epsilon \leq \psi_n(x) \leq f(x) + \epsilon, \quad \forall\, n \geq n_0.$$

$$\left.\begin{array}{l} \Rightarrow \quad \psi_n(x) \longrightarrow f(x) \\ \text{Similarly } \varphi_n(x) \longrightarrow f(x) \end{array}\right\} \forall\, x \in I \setminus D.$$

But $m \leq \psi_n(x) \leq f(x) \leq \varphi_n(x) \leq M$, then by BCT, we get

$$\int_I f\,dm = \lim \int_I \varphi_n\,dm = \lim U(P_n, f)$$

$$\& \quad \int_I f\,dm = \lim \int_I \psi_n\,dm = \lim L(P_n, f)$$

Hence, $f \in R[a,b]$.

Remark: We have observed that $C[a,b] \subsetneq R[a,b] \subsetneq L^1[a,b]$. However, we will latter show that $\overline{C[a,b]} = L^1[a,b]$ and hence $\overline{R[a,b]} = L^1[a,b]$. Thus, a Lebesgue integral function is limit of R-int. functions on $[a,b]$.

Improper Riemann Integration:

Let $f:[a,\infty) \longrightarrow \mathbb{R}$ be such that

(i) $f \in R[a,b]$, $\forall\ b > a$.

(ii) $\lim\limits_{b\to\infty} \int_a^b f(x)\,dx$ exists (finite).

Then we say that f is improper R-int and its improper integral

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

Remark: An improper R-integrable function need not be Lebesgue integrable.

Example: Consider

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx.$$

Since $\frac{\sin x}{x}$ is bounded on $[0,1]$, if we write

$$g(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}, \text{ then } g \in R[0,1].$$

For $a > 1$, $$\int_1^a \frac{\sin x}{x}\,dx = \left[-\frac{\cos x}{x}\right]_1^a - \int_1^a \frac{\cos x}{x^2}\,dx$$

$$= \cos 1 - \frac{\cos a}{a} - \int_1^a \frac{\cos x}{x^2}\,dx$$

$$\lim_{a\to\infty} \int_1^a \frac{\sin x}{x}\,dx = \cos 1 - 0 - \lim_{a\to\infty} \int_1^a \frac{\cos x}{x^2}\,dx \text{ is finite.}$$

Hence $f \in IR[0,\infty)$. Further,

$$\int_{[0,\infty)} |f|\,dm = \sum_{n=1}^{\infty} \int_{[(n-1)\pi,\, n\pi)} |f|\,dm = \sum_{n=1}^{\infty} \int_{(n-1)\pi}^{n\pi} \left|\frac{\sin x}{x}\right| dx$$

(by Beppo-Levi thm)

$$\int_{[0,\infty)} |f|\,dm \geq \sum_{n=1}^{\infty} \frac{1}{n\pi} \int_{(n-1)\pi}^{n\pi} |\sin x|\,dx = \sum_{n=1}^{\infty} \frac{1}{n\pi} \int_0^{\pi} \sin x\,dx = \sum \frac{2}{n\pi} = \infty.$$

(Put $t = x - (n-1)\pi$ etc.) 134

Hence $f \notin L^1[0,\infty)$.

Ex. Show that $\sin x$ on $[0,\infty)$ (or $\mathbb{R}$) is not improperly R-integrable.

<u>Theorem</u>: Let $f : [a,\infty) \longrightarrow \mathbb{R}$ be such that

(i) $f \in R[a,b]$, $\forall\, b > a$,

(ii) $\int_a^b |f(x)|\,dx \leq M$, $\forall\, b > a$ (M is independent of b).

Then both f & $|f|$ are improper R-int. on $[a,\infty)$ and $f \in L^1[a,\infty)$ with

$$\int_{[a,\infty)} f\,dm = \int_a^{\infty} f(x)\,dx.$$

Proof: Since $\int_a^n |f(x)|\,dx$ $\uparrow$ sequence & bdd above,

$\lim_{n\to\infty} \int_a^n |f(x)|\,dx < \infty$. Hence $f \in IR[a,\infty)$.

Now, $0 \leq |f(x)| - f(x) \leq 2|f(x)|$, we get

$$\int_a^b (|f(x)| - f(x))\,dx \leq 2\int_a^b |f(x)|\,dx \leq 2M.$$ By previous case, $|f| - f \in IR[a,\infty)$. Hence, $f \in IR[a,\infty)$, because $f = (f - |f|) + |f|$.

Further, $f \in R[a,b] \Rightarrow f \in L^1[a,b], \ \forall\, b > a$ and hence $\int_{[a,b]} |f| dm \le M, \ \forall\, b > a$. By previous Theorem on page 129, it follows that

$f \in L^1[a,\infty)$ and

$$\int_{[a,\infty)} f\,dm = \lim_{b\to\infty} \int_{[a,b)} f\,dm = \lim_{b\to\infty} \int_a^b f(x)\,dx = \int_a^\infty f(x)\,dx.$$

Ex. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1+x^2}$.

Then $\int_a^b |f(x)|\,dx = \tan^{-1} b - \tan^{-1} a \le \pi, \ \forall\, a,b \in \mathbb{R}$

Then $f \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} f\,dm = \lim_{\substack{a\to -\infty \\ b\to\infty}} \int_a^b f(x)\,dx = \pi.$$

<u>Theorem</u>: Let $f: [0,a] \longrightarrow \mathbb{R}$ be such that

(i) $f \in L^1[\epsilon, a], \ \forall\, \epsilon > 0$ &

(ii) $\int_{[\epsilon,a]} |f|\,dm \le M, \ \forall\, \epsilon > 0$.

Then $f \in L^1[0,a]$ and $\int_{[0,a]} f\,dm = \lim_{\epsilon\to 0} \int_{[\epsilon,a]} f\,dm$.

<u>Proof</u>: Let $f_n = \chi_{(1/n,a]} f$. Then $f_n \xrightarrow{p.w.} f$ &

$|f_n| \uparrow |f|$ p.w. By MCT,

$$\int_{[0,a]} |f|\,dm = \lim_{n\to\infty} \int |f_n|\,dm < \infty. \text{ Hence}$$

$f \in L^1[0,a]$. Now, $|f_n| \le |f| \in L^1[0,a]$, and

$f_n \to f$ p.w., by DCT, $\int_{[0,a]} f\,dm = \lim_{n\to\infty} \int_{[1/n,a]} f$.

Theorem: Let $f : [a,\infty) \to \mathbb{R}$ be such that $f \in R[a,b]$, $\forall\, b > a$. Then $f \in L^1[a,\infty)$ iff $|f|$ is improper R-integrable. 136

Proof: Let $f \in L^1[a,\infty)$. Then $f_n = \chi_{[a,n)} f$ converges p.w. to f, and $|f_n| < |f| \in L^1$.

By DCT,
$$\infty > \int_{[a,\infty)} |f|\, dm = \lim \int |f_n| = \lim \int_a^n |f(x)|\, dx$$
$$= \int_a^\infty |f(x)|\, dx.$$

i.e. $f \in R[a,\infty)$.

Conversely, suppose $|f| \in IR[a,\infty)$. Then for $g_n = \chi_{[a,n)} |f|$, $g_n \uparrow |f|$. By MCT,
$$\int |f|\, dm = \lim \int_{[a,n)} |g_n|\, dm = \lim \int_a^n |f(x)|\, dx = \int_a^\infty |f(x)|\, dx$$

Hence $f \in L^1[a,\infty)$.

L^p-spaces:

Let (X, S, μ) be a measure space. For $1 \le p < \infty$, we write
$$L^p(X,S,\mu) = \{ f : X \xrightarrow{\text{m'ble}} \overline{\mathbb{R}} \text{ s.t. } \int_X |f|^p\, d\mu < \infty \}.$$

Then L^p is a linear space by identifying
$$[0] = \{ g \in L^p : g = 0 \text{ a.e. on } X \}.$$

Let $f, g \in L^p(X,S,\mu)$. Then

$|f+g|^p \le (|f|+|g|)^p \le \{2\max\{|f|,|g|\}\}^p$

$$\le 2^p \begin{cases} |f|^p & \text{if } |f|\ge|g| \\ |g|^p & \text{if } |f|\le|g| \end{cases}$$

$$\le 2^p(|f|^p+|g|^p).$$

Hence $\int|f+g|^p \le 2^p\int|f|^p + 2^p\int|g|^p < \infty$.

i.e. $f+g\in L^p$.

In general, $L^1 \not\subseteq L^2$ and $L^2 \not\subseteq L^1$.

For this, let $f(x) = \frac{1}{\sqrt{x}}\chi_{(0,1]}(x)$. Then $f\in L^1(\mathbb{R})$ but $f\notin L^2(\mathbb{R})$. Again, $g(x) = \frac{1}{1+|x|}$, $x\in\mathbb{R}$, $g\in L^2(\mathbb{R})$ but $g\notin L^1(\mathbb{R})$.

$$\int_{\mathbb{R}}|f|\,dm = 2\int_{[0,\infty]}\frac{1}{1+x}\,dm = \sum_{n=1}^{\infty}\int_{n-1}^{n}\frac{1}{1+x}\,dx \ge \sum_{n=1}^{\infty}\frac{1}{1+n} = \infty.$$

Ex. Let $f = \frac{1}{\sqrt{x}}\chi_{(0,1]}$ and write $f_n(x) = f(|x-n|)$.

Define $g = \sum \frac{1}{2^n} f_n$. Then $g\in L^1(\mathbb{R})$ but $g\notin L^2(\mathbb{R})$. For this consider

$$\int_{\mathbb{R}} g\,dm = \sum_{n=0}^{\infty}\frac{1}{2^n}\int f_n\,dm = \sum_{n=0}^{\infty}\frac{1}{2^n}\int_{\mathbb{R}}\frac{1}{\sqrt{|x-n|}}\chi_{[n,n+1]}(x)\,dm$$

$$= \sum\frac{1}{2^n}\int_{(0,1]}\frac{1}{\sqrt{x}}\,dm = \sum_{n=0}^{\infty}\frac{1}{2^n}\cdot 2 = 4.$$

Now, $\int_{\mathbb{R}} g^2\,dm = \sum\frac{1}{2^{2n}}\int_{\mathbb{R}}|f_n|^2\,dm = \sum\frac{1}{2^{2n}}\int_0^1\frac{1}{x}\,dm =$

(Hint: use the fact that if $E_1\cap E_2 = \phi$, then χ_{E_1} & χ_{E_2} are linearly independent)