Properties of m^*:

(i) If $A \subset B \subset \mathbb{R}$, then $m^*(A) \le m^*(B)$.

let $B \subset \bigcup I_n$, then $A \subset \bigcup I_n$. By def^n:

$m^*(A) \le \sum_{n=1}^{\infty} \ell(I_n)$ $\forall$ cover $\{I_n\}_{n=1}^{\infty}$ of B.

$\Rightarrow m^*(A) \le m^*(B)$. (i.e. m^* is monotone on $P(\mathbb{R})$)

(ii) If $\{A_n\}$ is a sequence of sets in $\mathbb{R}$.

then $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^*(A_n)$. $\boxed{\text{$m^*$ is countably subadditive on $P(\mathbb{R})$}}$

By def^n of infimum, for $\epsilon > 0$, $\exists$ a cover $\{I_{n,k}\}_{k=1}^{\infty}$ of A_n s.t.

$$\sum_{k=1}^{\infty} \ell(I_{n,k}) < m^*(A_n) + \frac{\epsilon}{2^n}$$

Thus, $\{I_{n,k} : n, k \in \mathbb{N}\}$ forms a cover of $\bigcup A_n$. Therefore

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \ell(I_{n,k})$$

$$\le \sum_{n=1}^{\infty} \left(m^*(A_n) + \frac{\epsilon}{2^n}\right)$$

$$\le \sum m^*(A_n) + \epsilon, \quad \forall \epsilon > 0.$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^*(A_n).$$

Ex. If $A \subset \mathbb{R}$ is countable, then

$A = \{a_1, a_2, \dots\} = \bigcup_{n=1}^{\infty} \{a_n\}$.

$\Rightarrow 0 \leq m^*(A) \leq \sum m^*(\{a_n\}) = 0 \Longrightarrow m^*(A) = 0.$

Thus, $m^*(\mathbb{Q}) = 0.$ (20)

Ex. Show that $m^*(\mathbb{R} \setminus \mathbb{Q}) = \infty.$

(Hint: $(-n, n) \subset \mathbb{R} \Rightarrow 2n \leq m^*(\mathbb{R})$, true)

Once again, if $\mathbb{Q} = \{ r_n : n \in \mathbb{Z} \}$, then

$$\mathbb{Q} \subset \left(r_n - \frac{\epsilon}{2^{|n|+1}}, r_n + \frac{\epsilon}{2^{|n|+1}} \right) \quad \text{—} (*)$$

Then $m^*(\mathbb{Q}) \leq \sum_{n \in \mathbb{Z}} \frac{\epsilon}{2^{|n|}} = 3\epsilon$, $\forall \epsilon > 0.$

$$\Rightarrow m^*(\mathbb{Q}) = 0.$$

Notice that RHS of (*) is an unbounded open set having outer measure as small as one desire, is a surprising fact about an "unbounded open" set.

<u>Result</u>: If I is any interval with end pts a & b. Then $m^*(I) = b - a.$

<u>Proof</u>: We first assume the result is true for closed interval $I = [a, b]$. That is

$$m^*([a, b]) = b - a.$$

If $I = (a, b)$, then $[a + \epsilon/2, b - \epsilon/2] \subset (a, b)$.

$$\Rightarrow b - a - \epsilon < m^*\{(a, b)\}, \ \forall \epsilon > 0.$$

$\Rightarrow$ $b-a \geq m^*\{(a,b)\}$.

Since (a,b) is a cover of itself,

$$m^*\{(a,b)\} \leq b-a.$$

$$\Rightarrow m^*\{(a,b)\} = b-a.$$

For other interval $(a,b]$ & $[a,b)$ one can do in a similar way.

$$(a,b) \subset (a,b] \subset [a,b]$$

$$\Rightarrow m^*\{(a,b]\} = b-a.$$

Now, we consider the case of proving

$$m^*([a,b]) = b-a.$$

$\therefore$ $[a,b] \subset (a-\frac{1}{n}, b+\frac{1}{n})$, $n \in \mathbb{N}$

$$\Rightarrow m^*([a,b]) \leq b-a+\frac{2}{n} \to b-a.$$

On the other hand, suppose, $[a,b] \subset \bigcup_{n=1}^{\infty} I_n$.

Then $[a,b] \subset \bigcup_{n=1}^{k} I_n$ (~~Exercise~~)

(Hint: use Bolzano-Weierstrass thm)

$\Rightarrow$ $(a,b) \subset \bigcup_{n=1}^{k} I_n$. By induction on k, and (a,b) being an interval, ~~(a,b)~~ it can be shown that

$$b-a \leq \sum_{n=1}^{k} l(I_n).$$

(Hint: if $(a,b) \subset \bigcup_{n=1}^{k} I_n \cup I_{k+1}$. Then $(a,b) \subset \bigcup_{n=1}^{k} I_n$ & $(a,b) \subset I_{n+1}$)

Thus, $b-a \le \sum_{n=1}^{k} l(I_n) \le \sum_{n=1}^{\infty} l(I_n)$, $\forall$ cover $\{I_n\}_{n=1}^{\infty}$ of $[a,b]$. Hence,

$$b-a \le m^*\{[a,b]\} \le b-a.$$

Ex. Let $A \subset \mathbb{R}$ & $x \in \mathbb{R}$ Then for $A+x = \{a+x : a \in A\}$, we have

$$m^*(A+x) = m^*(A).$$

Let $A \subset \bigcup I_n$. Then $A+x \subset \bigcup I_n + x = \bigcup (I_n + x)$

$$\Rightarrow m^*(A+x) \le \sum l(I_n + x) = \sum l(I_n).$$

Since $\{I_n\}$ is an arbitrary cover of A, it follows that

$$m^*(A+x) \le m^*(A). \quad \text{---(1)}$$

By replacing $x \to -x$ & then $A \to A+x$, it follows that

$$m^*(A-x) \le m^*(A) \text{ and}$$

then $m^*(A) \le m^*(A+x)$ ---(2)

By (1) & (2), $m^*(A+x) = m^*(A)$.

i.e. m^* is translation invariant.

Ex. Let $A \subset \mathbb{R}$ be such that $m^*(A) < \infty$. Then for each $\epsilon > 0$, $\exists$ an open set $O \supset A$ s.t. $m^*(O) < m^*(A) + \epsilon$. (23)

Proof: Since $m^*(A) < \infty$. For $\epsilon > 0$, $\exists$ a cover $\{I_n\}$ of A s.t.

$$\sum l(I_n) < m^*(A) + \epsilon. \quad \cancel{\text{But } m^*(\cup I_n)}$$

Let $O = \cup I_n$. Then $m^*(O) \leq \sum l(I_n) < m^*(A) + \epsilon$.

(by countable subadditivity of m^*).

Result: If $A \subseteq \mathbb{R}$, then $m^*(A) = \inf\{m^*(O) : O \supset A, O \text{ open}\}$.

proof: If $m^*(A) = \infty$, then for any $O \supset A$, $m^*(O) = \infty$. Thus, the result is true.

Suppose $m^*(A) < \infty$. Then by previous ~~exercise~~ exercise, for each $\epsilon > 0$, $\exists$ an open set $O \supset A$ s.t.

$$m^*(O) < m^*(A) + \epsilon.$$

But $m^*(A) \leq m^*(O)$. Thus.

$$m^*(A) = \inf\{m^*(O) : O \supset A, O\text{-open}\}.$$

Result: If $A \subset \mathbb{R}$, then $\exists$ a G_δ-set $G \supset A$ s.t. $m^*(A) = m^*(G)$. ——— $(*)$

Proof: If $m^*(A) = \infty$, then $(*)$ is true.

Let $m^*(A) < \infty$, then by previous exercise, for each $n \in \mathbb{N}$ $(\epsilon = \frac{1}{n} > 0)$, $\exists$ open set $O_n \supset A$ st

(24)

$$m^*(O_n) < m^*(A) + \frac{1}{n}.$$

Write $G = \bigcap_{n=1}^{\infty} O_n$. Then

$$m^*(G) \leq m^*(O_n) < m^*(A) + \frac{1}{n}$$

i.e. $m^*(G) < m^*(A) + \frac{1}{n}$, $\forall n \geq 1$.

$$\Rightarrow m^*(G) \leq m^*(A) \leq m^*(G).$$

(by monotone)

Uniqueness property of set of outer measure zero.

Ex. Let $E = \bigcup E_n$, $E_n \subset \mathbb{R}$. Then $m^*(E) = 0$ iff $m^*(E_n) = 0$, $\forall n \geq 1$.

Sf $m^*(E_n) = 0$, $\forall n \geq 1$, by countable sub-additivity $m^*(E) \leq \sum m^*(E_n) = 0$.

On the other hand, if $m^*(E) = 0$. Then

$$m^*(E_n) \leq m^*(E) = 0 \text{ (by monotone)}$$

If we want to skip the monotone property, then ~~can~~ $\Rightarrow$ the forward implication can be done in a nice way.

Suppose $m^*(E) = 0$ & $m^*(E_{n_0}) > 0$ for some $n_0 \in \mathbb{N}$. Then for $\epsilon = \frac{1}{2} m^*(E_{n_0}) > 0$, $\exists$ a cover $\{I_k\}$ of E s.t.

$$\sum l(I_k) < m^*(E) + \frac{1}{2} m^*(E_{n_0})$$

But $E_{n_0} \subset E \subset \cup I_k \Rightarrow m^*(E_{n_0}) < \sum l(I_k)$

$$\Rightarrow \quad m^*(E_{n_0}) < \frac{1}{2} m^*(E_{n_0}).$$

Ex. Let $O = \cup I_n$, I_n - open intervals. Then

$$m^*(O) = \sum l(I_n).$$

Proof: For $\epsilon > 0$, $\exists$ a cover $\{J_k\}$ of O s.t.

$$\sum l(J_k) < m^*(O) + \epsilon \quad \text{——} (1)$$

Now, $\cup I_n = O \subset \cup J_k$. Given I_n's are disjoint, each $I_n \subset J_{k,n}$, ~~whe~~ for some $J_{k,n} \in \{J_k\}$. ($\because$ these J_k's could also be considered disjoint)

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \le \sum_{n=1}^{\infty} l(J_{k,n}) \le \sum_{k=1}^{\infty} l(J_k) < m^*(O) + \epsilon$$

i.e. $\quad \sum l(I_n) < m^*(O) + \epsilon, \ \forall \ \epsilon > 0.$

Thus, $\sum l(I_n) \le m^*(O) \le m^*(\cup I_n) \le \sum m^*(I_n)$.

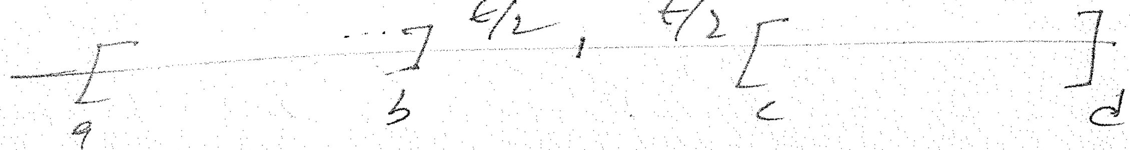
Remark: This shows that m^* is countably additive on ~~$P(\mathbb{R})$~~. collection of all open intervals.

Result: Let $[a,b] \cap [c,d] = \emptyset$. Then

$$m^*([a,b] \cup [c,d]) = m^*([a,b]) + m^*([c,d]).$$

Proof: Since $[a,b] \cap [c,d] = \emptyset$, we may assume $[a,b]$ & $[c,d]$ are separated by distance say $\epsilon > 0$.

Suppose $[a,b] \cup [c,d] \subset \cup I_n$. Then



then $[a,b] \subset \cup (I_n \cap (a-\epsilon/2, b+\epsilon/2)) = \cup I_n'$ (say)

& $[c,d] \subset \cup (I_n \cap (c-\epsilon/2, d+\epsilon/2)) = \cup I_n''$ (say)

Then $I_n' \cap I_l' = \emptyset$ $\forall\, n, l \in \mathbb{N}$.

$$\Rightarrow m^*([a,b]) + m^*([c,d]) \le \sum l(I_n') + \sum l(I_n'')$$

$$= \sum l(I_n' \cup I_n'')$$

$$= \sum l\{I_n \cap ((a-\epsilon/2, b+\epsilon/2) \cup (c-\epsilon/2, d+\epsilon/2))\}$$

$$\le \sum l(I_n), \quad \forall \text{ cover } \{I_n\} \text{ of } [a,b] \cup [c,d].$$

Thus, $m^*([a,b]) + m^*([c,d]) \le m^*([a,b] \cup [c,d])$.

Since m^* is sub-additive, the other inequality holds. (27)

<u>Remark</u>: If $A, B \subset \mathbb{R}$ & $\text{dist}(A, B) = \epsilon > 0$, then $m^*(A \cup B) = m^*(A) + m^*(B)$.

(Hint: If either $m^*(A) = \infty$ or $m^*(B) = \infty$, then result holds. If $m^*(A), m^*(B) < \infty$, then, still we have two cases, when A & B bounded & while A & B are not bounded. Hence, I leave it as exercise to students to complete).

Ex - Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of disjoint cpt sets in $\mathbb{R}$. Then

$$m^*\left(\dot{\bigcup_{n=1}^{\infty}} K_n\right) = \sum_{n=1}^{\infty} m^*(K_n). \quad \text{—} (*)$$

<u>Proof</u>: By the previous result.

$$\sum_{n=1}^{l} m^*(K_n) = m^*\left(\dot{\bigcup_{n=1}^{l}} K_n\right) \leq m^*\left(\dot{\bigcup_{n=1}^{\infty}} K_n\right). \text{—} (**)$$

Let $F = \dot{\bigcup_{n=1}^{\infty}} K_n$. If $m^*(F) = \infty$, then $(*)$ will holds by countable subadditivity.

Now, ~~let~~ $m^*(F) < \infty$. Then by $(**)$, we get

$$\sum_{n=1}^{\infty} m^*(K_n) \leq m^*\left(\dot{\bigcup_{n=1}^{\infty}} K_n\right) \leq \sum_{n=1}^{\infty} m^*(K_n).$$

Thus, the collection of well-separated sets satisfies the countable additivity.

Question: What are all those sets for which m^* is countably additive i.e. $m^*(\cup E_n) = \sum m^*(E_n)$?

Next, we will try to characterize those sets via the following example.

Ex. Suppose G is an open & bounded subset of $\mathbb{R}$. Then for each $\epsilon > 0$, $\exists$ compact set $K \subset G$ s.t. $m^*(K) > m^*(G) - \epsilon$

proof: Since G is bdd, $G \subset [a,b]$ & hence $m^*(G) \leq b-a < \infty$.

Further, G is open, therefore,

$$G = \cup I_n \Rightarrow m^*(G) = \sum_{n=1}^{\infty} l(I_n) < \infty.$$

This implies, for $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t.

$$\sum_{n=N+1}^{\infty} l(I_n) < \epsilon/2 \quad \text{--- (1)}$$

Let $K = \bigcup_{n=1}^{N} \left[a_n + \frac{\epsilon}{4N}, b_n - \frac{\epsilon}{4N}\right]$, where $I_n = (a_n, b_n)$.

Then $m^*(K) = \sum_{n=1}^{N} m^*\left(\left[a_n + \frac{\epsilon}{4N}, b_n - \frac{\epsilon}{4N}\right]\right)$ ($\because$ each of them is cpt & disjoint to others)

$= \sum_{n=1}^{N}\left(l(I_n) - \frac{\epsilon}{2N}\right)$

$= \sum_{n=1}^{N} l(I_n) - \frac{\epsilon}{2}$

$\therefore m^*(K) = \sum_{n=1}^{N} l(I_n) + \epsilon/2 - \epsilon$

$> \sum_{n=1}^{N} l(I_n) + \sum_{n=N+1}^{\infty} l(I_n) - \epsilon$

ie $m^*(K) > m^*(G) - \epsilon$.

Now, $m^*(G \setminus K) \leq m^*\left(G \setminus \bigcup_{n=1}^{N} I_n\right) + m^*\left(\bigcup_{n=1}^{N} I_n \setminus K\right)$

$= \sum_{n=N+1}^{\infty} l(I_n) + \epsilon/2.$ (Exercise)

$\cancel{\leq m^*(G) + \sum_{n=1}^{N} l(I_n) + \epsilon/2}$

$= \sum_{n=N+1}^{\infty} l(I_n) + \sum_{n=1}^{N} l(I_n) - m^*(K)$ (Ex.)

$= m^*(G) - m^*(K) < \epsilon$ —— $(*)$

ie $\forall \epsilon > 0$, $\exists$ cpt set $K \subset G$ st

$m^*(G \setminus K) < \epsilon$.

Ex. Let G be open & bounded set in $\mathbb{R}$ & F be any closed set in G. Then

$m^*(G \setminus F) = m^*(G) - m^*(F)$.

Since $G \setminus F$ is open & bounded, $\forall \epsilon > 0$, $\exists$

Compact set $K \subset G \setminus F$ s.t. $m^*(G \setminus F) - m^*(K) < \epsilon$.

Now, (30)

$$m^*(G) \leq m^*(G \setminus F) + m^*(F)$$
$$< m^*(K) + m^*(F) + \epsilon$$

But $K \cap F = \phi$ & K, F are cpt. set

$$\Rightarrow m^*(G) \leq m^*(G \setminus F) + m^*(F) < m^*(K \cup F) + \epsilon$$
$$< m^*(G) + \epsilon$$

ie $m^*(G) \leq m^*(G \setminus F) + m^*(F) < m^*(G) + \epsilon$ $\forall \epsilon > 0$.

$$\Rightarrow m^*(G) = m^*(G \setminus F) + m^*(F).$$

Ex. If G is an open set in $\mathbb{R}$. Then for each $\epsilon > 0$, $\exists$ a closed set $F \subset G$ s.t.

$$m^*(G \setminus F) < \epsilon.$$

proof: let $m^*(G) = \infty$. Then

$$G = \bigcup_{n \in \mathbb{Z}} (G \cap (n, n+1]) = \bigcup G_n, \text{ where}$$

$$G = \bigcup_{n \in \mathbb{Z}} G_n = \bigcup G \cap (n, n+1] = \bigcup (G_n' \cup (G \cap \mathbb{Z})),$$

where $G_n' = G \cap (n, n+1)$, Hence, G_n' is open & bdd. Hence, for $\epsilon > 0$, $\exists$ cpt set $K_n \subset G_n'$ s.t. $m^*(G_n' \setminus K_n) < \frac{\epsilon}{2^{|n|}}$.

Let $F = \bigcup_{n=-\infty}^{\infty} K_n$. We show that F is closed. In this case, since $m^*(G_n) = m^*(G_n')$, we have

$$m^*(G \setminus F) = m^*\Big(\bigcup_{n \in \mathbb{Z}} (G_n \setminus K_n)\Big) \quad (\text{Ex.})$$

$$\leq \sum m^*(G_n \setminus K_n) = \sum m^*(G_n' \setminus K_n) < 3\epsilon.$$

$$(\because \; m^*(A \cup B) = m^*(B) \text{ if } m^*(A) = 0.)$$

Now, let $x_k \in F = \bigcup K_n$, $K_n \subset G_n' = G \cap (n, n+1)$, and $x_k \to x$. Then for $\epsilon = 1 > 0$, $\exists$ $K_0 \in \mathbb{N}$ s.t. $k \geq K_0 \Rightarrow x_k \in (x-1, x+1) \subset (n-1, n+1)$.

Since $x_k \in \bigcup K_n \Rightarrow x_k \in K_{n-1} \cup K_n \subset F$ $\forall$ $k \geq K_0$. Thus, $x \in K_{n-1} \cup K_n \subset F$.

Lebesgue measurable sets:

A set $E \subseteq \mathbb{R}$ is said to be ~~Lebesgue~~ Lebesgue measurable (L-m'ble) if for each ~~...~~ $\epsilon > 0$, $\exists$ open set $O \supseteq E$ & closed set $F \subset E$ s.t.

$$m^*(O \setminus F) < \epsilon.$$

Note that $m^*(O \setminus E) < m^*(O \setminus F) < \epsilon$ & $m^*(E \setminus F) < m^*(O \setminus F) < \epsilon$.

Hence, L-mble sets are approximately open & closed. (32)

<u>Result</u>: Let $\mathcal{M}$ denote the collection of all L-mble sets in $\mathbb{R}$.

(i) If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$

$O^c \subset E^c \subset F^c$ & $m^*(F^c \setminus O^c) < \epsilon$.

(ii) If $m^*(E) = 0$, then $E \in \mathcal{M}$.

Since $m^*(E) = 0 < \infty$, for $\epsilon > 0$, $\exists$ open set $O \supset E$ st $m^*(O) < m^*(E) + \epsilon$.

Let $F \subset E$ be closed, then

$m^*(O \setminus F) \leq m^*(O) < \epsilon$.

(iii) $E_1, E_2 \in \mathcal{M}$, then $E_1 \cup E_2 \in \mathcal{M}$.

For $\epsilon > 0$, $\exists$ $F_i \subset E_i \subset O_i$; $i = 1, 2$.

st $m^*(O_i \setminus F_i) < \epsilon/2$, $i = 1, 2$.

For $O = O_1 \cup O_2$ & $F = F_1 \cup F_2$

$O \setminus F \subset \bigcup_{i=1}^{2} (O_i \setminus F_i)$ (Ex)

$\Rightarrow m^*(O \setminus F) < \epsilon/2 + \epsilon/2 = \epsilon$.