(iii) If $\{E_n\}_{n=1}^{\infty} \subset M$, then $\bigcup_{n=1}^{\infty} E_n \in M$.

For proving this, we need the following lemma.

Lemma: Let $E \subseteq \mathbb{R}$. Then $E \in M$ iff
$E \cap (a,b) \in M$, $\forall$ $[a,b] \in \mathbb{R}$.

Proof: If $E \in M$, then $E \cap (a,b) \in M$, since open sets are L-m'ble.

Conversely, suppose each $E \cap (a,b) \in M$, $a,b \in \mathbb{R}$. Then $E \cap (n, n+1] = (E \cap (n, n+1)) \cup N$,
when $N = E \cap \{n+1\}$ & $m^*(N) = 0$.
Hence $E \cap (n, n+1] \in M$. Now,
$$E = \bigcup_{n \in \mathbb{Z}} (E \cap (n, n+1]) = \bigcup_{n \in \mathbb{Z}} E_n.$$
Then $E_n = E \cap (n, n+1] \in M$. For $\epsilon > 0$, $\exists$
$F_n \subset E_n \subset O_n$ st $m^*(O_n \setminus F_n) < \frac{\epsilon}{2^{|n|}}$

~~Since~~ Let $O = \bigcup O_n$, & $F = \bigcup_{n \in \mathbb{Z}} F_n$. Then
F is closed. Thus,
$m^*(O \setminus F) \leq m^*(\bigcup (O_n \setminus F_n)) < \sum m^*(O_n \setminus F_n)$
ie $m^*(O \setminus F) < 3\epsilon$.

Let $E = \bigcup_{n=1}^{\infty} E_n$, $E_n \in M$. Let $E_n' = E_n \setminus \bigcup_{i=1}^{n-1} E_i$.
Then $E_n' \cap E_m' = \phi$, if $n \neq m$, & $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} E_n'$.

(If not: let $m<n \Rightarrow m\le n-1$. If $x\in E_n'\cap E_m'$; $x\in E_n'$ & $x\in E_n' = E_n\setminus\bigcup_{i=1}^{n-1}E_i \Rightarrow x\notin E_m'$)

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Hence, without loss of generality, we can assume $E=\dot{\bigcup}_{n=1}^{\infty}E_n$, $E_n\cap E_m=\phi$, if $n\neq m$.

Further, by the previous lemma, it is enough to prove the result for E bdd.

Thus, $m^*(E_n)\le m^*(E)<\infty$, & E_n's are bounded. Hence, for $\epsilon>0$, $\exists$ $F_n\subset E_n\subset O_n$ s.t. $m^*(O_n\setminus F_n)<\frac{\epsilon}{2^{n+1}}$ $\forall n\ge 1$.

Now,
$$\sum_{n=1}^{k}m^*(O_n)\le\sum_{n=1}^{k}m^*(O_n\setminus F_n)+\sum_{n=1}^{k}m^*(F_n)$$
$$<\sum_{n=1}^{k}\frac{\epsilon}{2^{n+1}}+m^*\Big(\dot{\bigcup_{n=1}^{k}}F_n\Big)\quad(\because F_n \text{ is closed \& bdd})$$
$$<\epsilon/2+m(E)<\infty,\ \forall k\ge 1.$$

i.e. $\sum_{n=1}^{\infty}m^*(O_n)<\infty$. For $\epsilon>0$, $\exists$ $N\in\mathbb{N}$ s.t. $\sum_{n=N+1}^{\infty}m^*(O_n)<\epsilon/2$.

Write $O=\bigcup_{n=1}^{\infty}O_n$ & $F=\dot{\bigcup}_{n=1}^{N}F_n$. Then O is open & F is closed.

$\therefore m^*(O\setminus F)=m^*\Big(\big(\bigcup_{n=1}^{N}O_n\big)\cup\big(\bigcup_{n=N+1}^{\infty}O_n\big)-F\Big)$

$$\Rightarrow m^*(O\setminus F) \leq m^*\left(\bigcup_{n=1}^{N} O_n\setminus F\right) + m^*\left(\bigcup_{n=N+1}^{\infty} O_n\setminus F\right)$$

$$(\because A\cup B\setminus C = (A\setminus C)\cup(B\setminus C))$$

$$< \sum_{n=1}^{N}\frac{\epsilon}{2^{n+1}} + m^*\left(\bigcup_{n=N+1}^{\infty} O_n\right)$$

$$\leq \epsilon/2 + \sum_{n=N+1}^{\infty} m^*(O_n) < \epsilon.$$

Ex. Let $E\in\mathcal{M}$, then $E+x\in\mathcal{M}$, $\forall x\in\mathbb{R}$.

For $\epsilon>0$, $\exists$ $O\supset E\supset F$ s.t.

$$m^*(O\setminus F)<\epsilon.$$

$F+x$ - closed & $O+x = \bigcup(I_n+x)$ is open. Thus, $F+x\subset E+x\subset O+x$.

$$\Rightarrow m^*(O+x\setminus(F+x))<\epsilon.$$

Ex. Verify that

(i) $(F+x)^c = F^c+x$.

(ii) $(O+x)\cap(F+x)^c = O\cap F^c+x$.

(Hint: $z\notin F+x \Rightarrow z-x\notin F \Rightarrow z-x\in F^c$
$\Rightarrow z\in F^c+x$ etc).

<u>Result</u>: Let $E=\dot{\bigcup}_{n=1}^{\infty}E_n$, $E_n\in\mathcal{M}$. Then

$$m^*\left(\dot{\bigcup}_{n=1}^{\infty}E_n\right) = \sum_{n=1}^{\infty}m^*(E_n).$$

Proof: (i) Suppose E is bounded. Then

$$m^*(E_n)\leq m^*(E)<\infty.$$

Since $E_n\in\mathcal{M}$, for $\epsilon>0$, $\exists$ $F_n\subset E_n\subset O_n$

st $\quad m^*(O_n\setminus F_n)<\frac{\epsilon}{2^n}.$

Since $\sum_{n=1}^{\infty} m^*(E_n) < \infty$, to prove $\sum_{n=1}^{\infty} m^*(E_n) \leq m(E)$, it is enough to

Consider $\sum_{n=1}^{k} m^*(E_n) \leq \sum_{n=1}^{k} m^*((O_n \setminus F_n) \cup F_n))$

$$\leq \sum_{n=1}^{k} m^*(O_n \setminus F_n) + \sum_{n=1}^{k} m^*(F_n)$$

$$< \sum_{n=1}^{k} \frac{\epsilon}{2^n} + m^*\left(\bigcup_{n=1}^{k} F_n\right) \quad (\because F_n \text{ is cpt})$$

$$< \epsilon + m^*(E) < \infty, \ \forall k \geq 1$$

$$\Rightarrow \sum_{n=1}^{\infty} m^*(E_n) \leq \epsilon + m^*(E), \ \forall \epsilon > 0.$$

$$\Rightarrow \sum_{n=1}^{\infty} m^*(E_n) \leq m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n).$$

Next, suppose E is <u>not</u> bounded. Then $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} (k, k+1]$. Let $A_k = E \cap (k, k+1]$ and $E_{n,k} = E_n \cap (k, k+1]$. Then

$$E = \bigcup_{k \in \mathbb{Z}} A_k \quad \text{and} \quad E_n = \bigcup_{k \in \mathbb{Z}} E_{n,k}.$$

Then $\sum_{n=1}^{\infty} m^*(E_n) \leq \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} m^*(E_{n,k})$ — (1)

Since $A_k = \bigcup_{n=1}^{\infty} E_{n,k}$, and A_k is bdd, by the previous case,

$$m^*(A_k) = \sum_{n=1}^{\infty} m^*(E_{n,k}) \quad \text{— (2)}$$

From (1) & (2), $\sum_{n=1}^{\infty} m^*(E_n) \le \sum_{k=-\infty}^{\infty} m^*(A_k)$ —— (3)

Now $\sum_{k=-l}^{l} m^*(A_k) = m^*\left(\bigcup_{k=-l}^{l} A_k\right) \le m^*(E)$ —— (4)

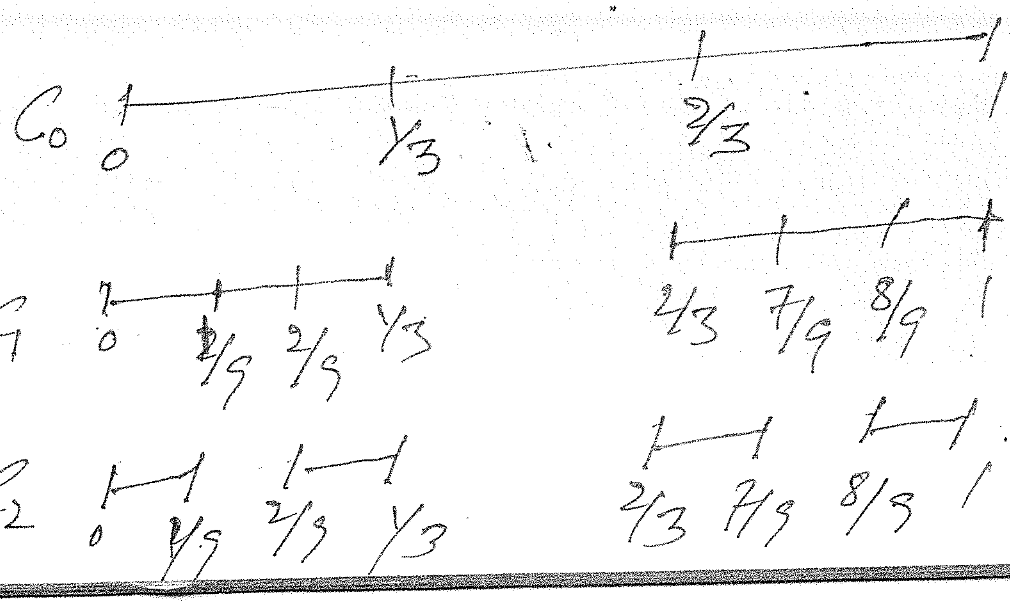
If $m^*(E) = \infty$, then the identity holds trivially, as $m^*(E) \le \sum_{n=1}^{\infty} m^*(E_n)$. 37

Let $m^*(E) < \infty$. Then (3) & (4), we get

$$\sum_{n=1}^{\infty} m^*(E_n) \le \sum_{k=-\infty}^{\infty} m^*(A_k) \le m^*(E) \le \sum_{n=1}^{\infty} m^*(E_n).$$

<u>Cantor set</u>: Cantor set is an uncountable set in $[0,1]$ having outer L-measure zero together with many peculiar properties which answers many question relates to topology of real line. We will consider ~~see~~ some of them.

Let $C_0 = [0,1]$



$C_0 = [0,1]$, 1 closed interval, length $= 1$ (38)

$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$, 2 disjoint closed intervals, each of length $= 1/3$.

$C_2 = [0,\frac{1}{9}] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, consists of 4 disjoint closed intervals, each of length 3^{-2}.

By induction, C_n ~~contains~~ made of 2^n intervals each having length $= 3^{-n}$.

(i) $\{C_n\}$ is a decreasing sequence of closed 4bdd sets, hence $C_n \in M$.

(ii) Let $C = \bigcap_{n=0}^{\infty} C_n$. then C contains all the end points of the deleted open intervals.

(iii) $C = [0,1] \setminus \{(\frac{1}{3},\frac{2}{3}) \cup (\frac{1}{9},\frac{2}{9}) \cup (\frac{7}{9},\frac{8}{9}) \cdots\}$

(iv) Since $C \subset C_n$, $\forall\ n \geq 0$.

$$m^*(C) \leq m^*(C_n) = 2^n \cdot \frac{1}{3^n} \longrightarrow 0.$$

(v) the Cantor ternary set C (later we just say Cantor set) is nowhere dense i.e. $(\bar{C})^{\circ} = C^{\circ} = \phi$. If not,

then $x \in C^{\circ} \Rightarrow (x-\epsilon, x+\epsilon) \subset C^{\circ} \subset C$.

$\Rightarrow 2\epsilon \leq m^*(C) = 0, \quad \epsilon > 0.$

which is a contradiction.

(vi) C is totally disconnected (i.e connected sets in C are only ~~singleton~~ singletons).

(vii) Every point of C is a limit point of C itself. (i.e C is a perfect set).

let $x \in C = \bigcap C_n \Rightarrow x \in C_n, \forall n \geq 0$.
Then x must belongs to one of the closed interval that constituence to C_n.
i.e $x \in [x_n, y_n]$. st $y_n - x_n = \frac{1}{3^n}$.

$\Rightarrow |x_n - x| < \frac{1}{3^n}, \forall n \geq 0 \Rightarrow x_n \to x$.

note that x_n & y_n are end pts of the deleted open intervals. Hence, $x_n, y_n \in C$.
Thus, if E denotes the set of all end points, then $\bar{E} = C$. Since E is countable, C is separable.
Hence, it is expected that C can be uncountable

(viii) Representation of Cantor set.

Consider the end pt $\frac{1}{3} \in C$. We can write $\frac{1}{3} = \frac{0}{3} + \frac{2}{3} + \frac{2}{3^2} + \cdots = (0.022\cdots)_3$

Similarly, $\frac{2}{3} = (0.2)_3$. In a similar way, it can be shown that all end pts expressed as $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots$, $a_i \in \{0, 2\}$.

Now, consider the set

$$F = \{x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i},\ a_i \in \{0,1,2\}\} \setminus \{\text{end points}\} = [0,1] \setminus E.$$

For $x \in F$, we have $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots$,

notice that $a_1 = 1$ iff $x \in (\frac{1}{3}, \frac{2}{3})$, iff $x \notin C_1$.

Further,

$$a_1 \neq 1,\ a_2 = 1 \text{ iff } x \in (\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9})$$
$$\text{iff } x \notin C_2.$$

Thus, $a_{i_0} = 1$ iff $x \in C_{i_0}$, for some i_0.

Since $C = \bigcap_{n=0}^{\infty} C_n$, we can show that

$$C = \{x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i},\ a_i \in \{0, 2\}\}.$$

Let $x \in C = \bigcap C_n$, and $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$. Suppose some of $a_{i_0} = 1$. Then $x \notin C_{i_0} \Rightarrow x \notin C$. (41)

Thus, all the $a_i \in \{0, 2\}$.

i.e. $C \subseteq \{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i},\ a_i \in \{0,2\} \}$. —— (*)

On the other hand, let $x \notin C$. Then $x \notin C_{i_0}$, for some i_0. That implies $a_{i_0} = 1$.

That is $x \notin$ R.H.S. of (*).

<u>Representation is unique</u>:

For every $x \in C$, $\exists !$ sequence $\{a_i\}$ from $\{0, 2\}$ such that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \quad \text{—— (1)}$$

Suppose $x = \sum_{i=1}^{\infty} \frac{b_i}{3^i} \longrightarrow (2)$. Then claim $a_i = b_i$ $\forall i$.

If not, let $a_{i_0} \neq b_{i_0}$ for some i_0.

Let i_0 be the smallest integer s.t. $a_{i_0} \neq b_{i_0}$. Then $a_i = b_i$, $i = 1, 2, \cdots, i_0 - 1$.

Hence, w.l.g., we can take $i_0 = 1$.

i.e. $a_1 \neq b_1 \Rightarrow a_1 = 0$ & $b_1 = 2$ (or otherwise)

Then from (1), $x \in [0, \frac{1}{3}]$ and from (2) $x \in [\frac{2}{3}, 1]$ which is a contradiction.

Exercise. Conclude without assuming $i_0 = 1$

Cantor set is uncountable:

Define $f : C \longrightarrow [0,1] = \{x = \sum_{i=1}^{\infty} \frac{a_i}{2^i} ; i \in \{0,1\}\}$

by $f(x) = f\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) = \sum_{i=1}^{\infty} \frac{(b_i/2)}{2^i}$, then

$b_i/2 \in \{0,1\}$, & $f(x) \in [0,1]$. Since each $x \in C$, has unique representation, the map f is well defined.

f is not one-one! binary rep. is not unique

$f(\frac{1}{3}) = f((0.022\cdots)_3) = (0.011\cdots)_2 = (0.1)_2 = \frac{1}{2}$

& $f(\frac{2}{3}) = f((0.2)_3) = (0.1)_2 = \frac{1}{2}$.

$\Rightarrow f(\frac{1}{3}) = f(\frac{2}{3})$.

Ex. Show that $f(x) = f(y)$ iff x, y are end points of one of the deleted open interval.

f is an onto map!

Let $f : C \longrightarrow [0,1] \ni y$

s.t. $f(x) = y = \sum_{j=1}^{\infty} b_j 2^{-j}$, let

$x = \sum \frac{2b_j}{3^j}$, then $f(x) = y$ holds.

Hence C is an uncountable set. (43)

f is monotone increasing:

Let $x, y \in C$ & $x < y$. Then $\exists$ the least positive integer $n \in \mathbb{N}$ such that $a_n < b_n$. Hence $a_i = b_i$; $i = 1, 2, \dots, n-1$.

Thus, while comparing $f(x)$ & $f(y)$, we can ignore 1st $n-1$ terms. Thus, w.l.g. we can assume ~~a_1~~ $n = 1$. That is,

$$a_1 < b_1 \Rightarrow a_1 = 0,\ b_1 = 2.$$

$$\left.\begin{array}{l} \therefore f(x) \le \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} \cdots = \frac{1}{2} \\ \& \ f(y) = \frac{1}{2} + \frac{b_2}{2^2} + \cdots \ge \frac{1}{2} \end{array}\right\} \Rightarrow f(x) \le f(y).$$

Notice that $f(1/3) = f(2/3) = 1/2$. Hence, by keeping f constant on each deleted interval, we can extend f ~~on~~ to $[0,1]$. Thus,

$$\tilde{f} : [0,1] \longrightarrow [0,1] \text{ st } \tilde{f}/_C = f \text{ is}$$

a monotone function which is onto. Hence, $\tilde{f}$ is continuous.

Ex. Let $f : [a,b] \longrightarrow [a,b]$ be onto & monotone. Show that f is continuous. (See Carother's book).

Hence $f : C \longrightarrow [0,1]$ is a conti onto map

We have shown that the class $\mathcal{M}$, the collection of all L-msble subsets of $\mathbb{R}$ is closed under countable union & complement and containing empty set. (44)

Such collection of sets is called σ-algebra of sets.

Let $\mathcal{B}(\mathbb{R})$ denote the collection of sets in $\mathbb{R}$ which are made of countable union of open sets and their complement.

i.e. $O_i \in \mathcal{J}_u = \{$all open sets in $\mathbb{R}\}$, then
$\bigcup_{i=1}^{\infty} O_i \in \mathcal{B}(\mathbb{R})$, $O_i^c \in \mathcal{B}(\mathbb{R})$, & $\mathcal{J}_u \subset \mathcal{B}(\mathbb{R})$.

Since each open set is made of countable intervals $\#(\mathcal{B}(\mathbb{R})) = 2^{\aleph} = c$

However, $C \in \mathcal{M}(\mathbb{R})$, is uncountable &

$$m^*(C) = 0 \Rightarrow P(C) \subset \mathcal{M}(\mathbb{R})$$

$$\Rightarrow \#(\mathcal{M}(\mathbb{R})) \geq 2^{c} = 2^{c}.$$

Thus, $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{M}(\mathbb{R})$.

We will see concrete example of L-msble set which is not Borel msble (i.e. Borel set) later, while discussing msble function.

However, any L-msble set differs with a Borel msble set on a null set (set of outer measure zero).

<u>Theorem</u>: Let E be a subset of $\mathbb{R}$. Then TFAE:

1. E is L-msble
2. For each $\epsilon > 0$, $\exists$ open set $O \supseteq E$ such that $m^*(O \setminus E) < \epsilon$.
3. For each $\epsilon > 0$, $\exists$ closed set $F \subseteq E$ such that $m^*(E \setminus F) < \epsilon$
4. $\exists$ an G_δ-set $G \supseteq E$ s.t. $m^*(G \setminus E) = 0$.
   (i.e. $E = G \setminus N$, $N = G \setminus E$, $m^*(N) = 0$)
5. $\exists$ an F_σ-set $F \subseteq E$ s.t. $m^*(E \setminus F) = 0$.
   (i.e. $E = F \cup N$, $N = E \setminus F$)

<u>Proof</u>: We prove $(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1)$ and $(1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (1)$.

$(1) \Rightarrow (2)$: Since $E \in \mathcal{M}$, ~~$\forall$~~ $\epsilon > 0$, $\exists$ open set O & closed set F s.t. $F \subseteq E \subseteq O$ and $m^*(O \setminus F) < \epsilon$.

$$\Rightarrow m^*(O \setminus E) \leq m^*(O \setminus F) < \epsilon.$$

$(2) \Rightarrow (4)$: For $\epsilon = 1/n > 0$, $\exists$ open set $O_n \supseteq E$

such that $m^*(O_n \setminus E) < \frac{1}{n}$. Let
$G = \bigcap_{n=1}^{\infty} O_n$. Then $m^*(G \setminus E) \leq m^*(O_n \setminus E)$
$\Rightarrow m^*(G \setminus E) < \frac{1}{n}, \ \forall n \geq 1$.
i.e. $m^*(G \setminus E) = 0 \Rightarrow G \setminus E \in M$.
$(4) \Rightarrow (1)$: $E = G \setminus (G \setminus E) \in M$.

$(1) \Rightarrow (3)$ is trivial. For $(3) \Rightarrow (5)$, let
$\epsilon = \frac{1}{n} > 0$, then $\exists$ closed set $F_n \subset E$ s.t.
$m^*(E \setminus F_n) < \frac{1}{n}$.
Write $F = \bigcup F_n$, then
$m^*(E \setminus F) \leq m^*(E \setminus F_n) < \frac{1}{n}, \forall n \geq 1$
$\Rightarrow m^*(E \setminus F) = 0 \Rightarrow E \setminus F \in M$.
$(5) \Rightarrow (1)$: $E = (E \setminus F) \cup F \in M$.

Non-measurable set:

Since $m^*(\mathbb{Q}) = 0$, while searching for non-Lebesgue measurable set, we need to ignore $\mathbb{Q}$.

For $x, y \in \mathbb{R}$, define $x \sim y$ iff $x - y \in \mathbb{Q}$. Then $\sim$ is an equivalence relation on $\mathbb{R}$. Hence it ~~fort~~ partions $\mathbb{R}$ into disjoint equivalent classes.

Let $x+Q = \{x+r : r \in Q\}$. Then $x+Q$ is an equivalence class under $\sim$. (47)

(i) $(x+Q) \cap [0,1] \neq \phi$, $\forall\, x \in \mathbb{R}$ (easy).

(ii) Let E be a subset of $[0,1]$, that contains exactly one member from each $x+Q$, where $x \in [0,1]$.

Let $Q \cap [-1,1] = \{r_1, r_2, \dots\}$ & write

$$E_i = E + r_i \; ; \; i = 1, 2, \dots$$

Then (iii) $E_i \cap E_j = \phi$, $\forall\, i \neq j$. For this,

let $z \in E_i \cap E_j$. Then $z = x + r_i = y + r_j$, where $x \neq y$, $x, y \in E$. Hence

$$x - y = r_j - r_i \in Q$$

$\Rightarrow$ $x \sim y$, is a contradiction to the defn of E, as E contains exactly one member from each $x+Q$.

(iv) $[0,1] \subset \bigcup_{i=1}^{\infty} E_i \subset [-1,2]$.

Let $x \in [0,1]$. Then by defn of E (see (ii)), $x+Q$ must contains a pt of E.

That is, $\exists\, ! \; y \in (x+Q) \cap E$.

$\Rightarrow$ $y - x \in Q \cap [-1,1]$. That is

$y - x = r_{i_0}$ $\Rightarrow$ $x = y - r_{i_0} \in E_{i_0}$.

(v) The set E is not L-m'ble. On contrary, if $E \in M$, then each of $E_i \in M$ and hence from (iv), (48)

$$1 \leq m^*\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) \leq 3.$$

$$\Rightarrow 1 \leq \sum_{n=1}^{\infty} m^*(E) \leq 3. \quad (\because m^*(E_i) = m^*(E))$$

which is not possible. Note that $m^*(E) > 0$. If not, then for $m^*(E) = 0$

$$\Rightarrow \quad m^*(E_i) = 0. \text{ As } [0,1] \subset \dot{\bigcup_{i=1}^{\infty}} E_i$$

$$\Rightarrow 1 \leq \sum m^*(E_i) = 0.$$

Remark (i) The L-outer measure m^* is not countably additive. For this, let $A = \dot{\bigcup_{i=1}^{\infty}} E_i$. Then $1 \leq m^*(A) \leq 3$.

But $\sum_{i=1}^{\infty} m^*(E_i) = \sum m^*(E) = \infty$.

$$\therefore \cancel{m^*(E)} \; m^*\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) \leq 3 < \infty = \sum_{i=1}^{\infty} m^*(E_i).$$

(ii) Whether m^* is finitely additive?

Suppose $m^*\left(\dot{\bigcup_{i=1}^{n}} A_i\right) = \sum_{i=1}^{n} m^*(A_i)$,

where $A_i \in P(\mathbb{R})$ = power set of $\mathbb{R}$.

(i.e. in other words, let m^* be finitely additive)

Then $n\, m^*(E) = \sum_{i=1}^{n} m^*(E_i) = m^*\left(\bigcup_{i=1}^{n} E_i\right) \leq 3.$

$\Rightarrow m^*(E) \leq \frac{3}{n}, \ \forall\, n \geq 1.$

$\Rightarrow m^*(E) = 0$, a contradiction.

$\therefore m^*$ is not finitely additive.

(iii) Every L-measurable subset of a non-L-measurable set has outer measure zero.
(i.e. $E \notin M$, $A \subset E$ & $A \in M \Rightarrow m^*(A) = 0$).

For this, let $A_i = A + r_i$, $r_i \in Q \cap [-1,1]$.

~~Since $m^*(A) = 0 \Rightarrow m^*(A_i) =$~~

Then $\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} E_i \subset [-1,2]$.

Since $A \in M \Rightarrow A_i \in M$. Hence,

$\sum_{i=1}^{n} m^*(A_i) = m^*\left(\bigcup_{i=1}^{n} A_i\right) \leq 3 \Rightarrow m^*(A) \leq \frac{3}{n}, \forall\, n \geq 1.$

$\Rightarrow m^*(A) = 0.$

(iv) Every L-m'ble set of +ve outer measure has non-measurable set.

~~i.e.~~ ~~If~~ i.e. if $E \in M(\mathbb{R})$, $m^*(E) > 0$
then $\exists\, F \subset E$ s.t. $F \notin M(\mathbb{R})$.

Proof: Proof is based on the fact that translate of a non-m'ble set is non-m'ble.

Firstly, let $E \subset [0,1]$ & $m^*(E) > 0$. (50)
Let N be a non-m'sble set in $[0,1]$.
write $\hat{N}_i = N + r_i$, $r_i \in Q$. Then
$\hat{N}_i \cap \hat{N}_j = \Phi$, if $i \neq j$ (easy)
and $\mathbb{R} = \bigcup_{i=1}^{\infty} \hat{N}_i$.
Let $x \in \mathbb{R}$, then $x \in [0,1] + r_k$, for some $r \in Q$.
$\Rightarrow x - r \in [0,1] \subset \bigcup_{i=1}^{\infty} N_i \subset [-1,2]$,
where $N_i = N + s_i$, $s_i \in Q \cap [-1,1]$.
$\Rightarrow x - r \in N_{i_0}$ for some i_0. But $r \in Q$
$x \in N_{i_0} + r = \hat{N}_{j_0}$, for some j_0.
Now, $E = \bigcup_{i=1}^{\infty} (E \cap \hat{N}_i)$. Then each of
$E \cap \hat{N}_i$ cannot be L-m'sble, because,
if $E \cap \hat{N}_i \subset \hat{N}_i$ is L-m'sble, $\forall i$,
then $m^*(E \cap \hat{N}_i) = 0$, $\forall i$
$\Rightarrow m^*(E) = 0$.
Hence, $\exists\, i_0$ st $E \cap \hat{N}_i \subset E$, $E \cap \hat{N}_i \notin M$.

Finally, let $E \in \mathbb{R}$ & $m^*(E) > 0$ &
$E \in M(\mathbb{R})$.

Then $m^*(E) = \sum_{n\in\mathbb{Z}} m^*(E\cap(n,n+1]) > 0.$

Hence, $\exists$ n_0 s.t. $m^*(E\cap(n_0,n_0+1]) > 0.$

Let $F = E\cap(n_0,n_0+1]$. Then

$\Rightarrow F-n_0 \subset (0,1]$ & $m^*(F-n_0) > 0.$

Hence, $\exists$ $H \subset F-n_0$ s.t. $H \notin M$.

$\Rightarrow H+n_0 \subset F \subset E$, with $H+n_0 \notin M$.

<u>Result</u>: Let $(E_n) \subset M$ be an $\uparrow$ seq^n of sets. Then $\lim_{n\to\infty} m^*(E_n) = m^*(\bigcup_{n=1}^{\infty} E_n)$.

Proof: Let $E = \bigcup_{n=1}^{\infty} E_n$. Then $m(E_n)\uparrow$ &

$\lim m(E_n) = \sup m(E_n) \leq m^*(E)$.

Now, $\bigcup_{n=1}^{\infty} E_n = E_1 \dot\cup \bigcup_{n=1}^{\infty} (E_{n+1}\setminus E_n)$

$\Rightarrow m^*(E) = m^*(E_1) + \sum_{n=1}^{\infty} m^*(E_{n+1}\setminus E_n)$

$= m^*(E_1) + \lim_{K\to\infty} \sum_{n=1}^{K} m^*(E_{n+1}\setminus E_n)$.

~~$= m^*\{E_1 \dot\cup (E$~~

$= \lim_{K\to\infty} \{ m^*(E_1) + \sum_{n=1}^{K} m^*(E_{n+1}\setminus E_n)\}$

$= \lim_{K\to\infty} \{ m^*\{ E_1 \dot\cup \bigcup_{n=1}^{K} (E_{n+1}\setminus E_n)\}\}$

$\Rightarrow m(E) = \lim_{k \to \infty} m^*(E_{k+1})$.

<u>Result</u>: Let $(E_n) \subset M$ be a $\downarrow$ seq^n s.t. $m^*(E_1) < \infty$. Then ~~$\lim(E_n) =$~~

$$\lim_{n\to\infty} m^*(E_n) = m^*\left(\bigcap_{n=1}^{\infty} E_n\right).$$

<u>Proof</u>: Since $m^*(E_n) \geq m^*(E_{n+1}) \geq m^*(\cap E_n)$.

$\Rightarrow \lim m^*(E_n) = \inf m^*(E_n) \geq m^*(\cap E_n)$.

$$E_1 \setminus \bigcap_{n=1}^{\infty} E_n = \dot{\bigcup_{n=1}^{\infty}} (E_n \setminus E_{n+1}) \quad (\text{Ex})$$

$$\Rightarrow m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m^*(E_n \setminus E_{n+1})$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} m^*(E_n \setminus E_{n+1})$$

$$= \lim_{k\to\infty} \left(m^*(E_1) - m^*(E_{k+1})\right)$$

Since $m^*(E_1) < \infty \Rightarrow$

$$m^*(\cap E_n) = \lim m^*(E_n).$$

<u>Alternative</u>: $E_1 \setminus E_n \uparrow$ seq^n, & use previous result.

Ex. Let $E_n = \mathbb{R} \setminus (-n, n)$, $n \in \mathbb{N}$. Then $E_n \downarrow$ & $\bigcap_{n=1}^{\infty} E_n = \phi \Rightarrow m^*(\cap E_n) = 0$.

But $m^*(E_n) = \infty$.