Inner regularity of Lebesgue measurable sets:

We shall show that every L-measurable sets can be approximated by compact sets in the set itself. Before proving this result, we need to work out the following lemma. (53)

Lemma: Let $E \in \mathcal{M}$ and $m^*(E) < \infty$. Then for each $\epsilon > 0$, $\exists$ a compact set $K \subset E$ such that $m^*(E \setminus K) < \epsilon$,

$$(\text{i.e. } m^*(E) < m^*(K) + \epsilon.)$$

Notice that $m^*(K) < m^*(E) < m^*(K) + \epsilon$.

$$\Rightarrow \quad m^*(E) = \sup\{m^*(K) : K \subset E\}.$$

Proof: Let $E = \bigcup_{n=1}^{\infty} (E \cap (-n, n)) = \bigcup_{n=1}^{\infty} E_n$.

Then $E_n = E \cap (-n, n)$ is an increasing sequence & hence $m^*(E) = \lim m^*(E_n)$.

Since $m^*(E) < \infty$, for $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $m^*(E) - m^*(E_N) < \frac{\epsilon}{2}$. (by def^n of conv. of seq^n) (1)

Further, $E_N \in \mathcal{M}$ & E_N is bounded. Hence, for each $\epsilon > 0$, $\exists$ closed set $K_N \subset E_N$ such that

$$m^*(E_N \setminus K_N) < \epsilon/2$$

$$\text{i.e.} \quad m^*(E_N) - m^*(K_N) < \epsilon/2. \quad (2)$$

Thus, from (1)&(2) $m^*(E) - m^*(K_N) < \epsilon$ where K_N is bounded & hence compact.

<u>Theorem:</u> Let $E \in \mathcal{M}$. Then

$$m^*(E) = \sup\{m^*(K) : K \overset{\text{cpt}}{\subset} E\}.$$

Proof: Case (i), if $m^*(E) = \infty$. In this case,

$$E = \bigcup (E \cap (-n,n)) = \bigcup E_n, \quad E_n \uparrow$$

& $\lim m^*(E_n) = m^*(E) = \infty$. —(*)

Since $m^*(E_n) < \infty$, by the previous lemma, $\forall \epsilon > 0$, $\exists$ CBF set $K_n \subset E_n$ s.t.

$$m^*(E_n) < m^*(K_n) + \epsilon.$$

Take $\epsilon = 1$. Then

$$m^*(E_n) - 1 < m^*(K_n) \leq \sup_{K \subset E} m^*(K).$$

By (*), $\infty = m^*(E) = \lim m^*(E_n) = \sup_{K \subset E} m^*(K)$.

Case (ii) if $m^*(E) < \infty$, then for each $\epsilon > 0$, $\exists$ a CBF set $K \subset E$ s.t.

$$m^*(E \setminus K) < \epsilon.$$

Hence $m^*(E) < m^*(K) + \epsilon$ —(1)

Notice that $m^*(K') \leq m^*(E)$, $\forall K' \subset E$,

$$\sup_{K' \subset E} m^*(K') \leq m^*(E) \quad —(2)$$

From (1) & (2), we get

$$m^*(E) = \sup\{m^*(K) : K \subset E,\ K \text{ is compact}\}.$$

Notice that if $E \in \mathcal{M}$, then we have shown that $m^*(E) = \inf\{m^*(O) : O \supseteq E,\ O\text{-open}\}$

$$= \sup\{m^*(K) : K \subset E,\ K \text{ cpt}\}.$$

Thus, if E is L-msble, we can say that $m^*(E)$ is a true length of the set $E \subset \mathbb{R}$.

Now onward, we write

$$m^*(E) = m(E), \text{ if } E \in \mathcal{M}.$$

The set function $m : \mathcal{M} \longrightarrow [0, \infty]$, which is countably additive is known as Lebesgue measure. The set function m is satisfying continuity like condition in the sense that $\forall \epsilon > 0$, $\exists$ open set $O \supseteq E$ s.t.

$$m^*(O \setminus E) < \epsilon.$$

Excercise. ~~For~~ $x \in \mathbb{R}$, write $f(x) = m\{E \cap (-\infty, x]\}$, where $E \in \mathcal{M}$ & $m(E) < \infty$. Then show that f is uniformly continuous on $\mathbb{R}$.

Caratheodory Criteria for measurability.

Let $E \in \mathcal{M}$ & E is bounded. Then $E \subset (a, b)$ for some $a < b < \infty$.

Notice that in this case, we have

$$m^*\{(a,b)\} = m^*(E) + m^*\{(a,b)\setminus E\}.$$

Let $I = (a,b)$. Then

$$(*) \text{ --- } m^*(I) = m^*(I \cap E) + m^*(I \setminus E).$$

We will soon see that I can be replaced by any subset $A \subseteq \mathbb{R}$. Thus, it is interesting question to see that if $(*)$ holds for a given set $E \subset \mathbb{R}$, does it imply, E is L-msble?

Theorem: $E \in \mathcal{M}$ iff $\forall A \subseteq \mathbb{R}$, satisfies

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E) \quad \text{--- (1)}$$

(Caratheodory Criteria)

Proof: Since $A = (A \cap E) \dot{\cup} (A \setminus E)$, for proving (1), it is enough to prove

$$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E).$$

Now, suppose $E \in \mathcal{M}$. If $m^*(A) = \infty$, then (1) is true. Let $m^*(A) < \infty$. Then $\exists$ a G_δ-set $G \supset E$ s.t. $m^*(A) = m^*(G) < \infty$.

$$\therefore m^*(A \cap E) + m^*(A \setminus E) \leq m^*(G \cap E) + m^*(G \setminus E).$$

Since $G = (G \cap E) \dot{\cup} (G \setminus E)$ & $G \cap E, G \setminus E \in \mathcal{M}$, it follows that

$$m^*(A\cap E) + m^*(A\setminus E) \leq m^*(G) = m^*(A).$$

Conversely, suppose that (1) holds for each $A \subseteq \mathbb{R}$. Claim: $E \in \mathcal{M}$.

Firstly, let $m^*(E) < \infty$. Then $\exists$ a G_δ-set $G \supseteq E$ s.t. $m^*(G) = m^*(E)$. Since (1) is true for all $A \subseteq \mathbb{R}$, take $A = G$, then

$$m^*(G) = m^*(G\cap E) + m^*(G\setminus E)$$
$$= m^*(E) + m^*(G\setminus E)$$

$\Rightarrow m^*(G\setminus E) = 0.$ ($\because m^*(G) = m^*(E) < \infty$).

Hence, $G\setminus E \in \mathcal{M}$. Thus, $E = G\setminus(G\setminus E) \in \mathcal{M}$.

If $m^*(E) = \infty$, then we can decompose

$$E = \dot{\bigcup_{n\in\mathbb{Z}}}(E\cap(n, n+1]) = \dot{\bigcup_{n\in\mathbb{Z}}} E_n.$$

To prove $E \in \mathcal{M}$, we need to prove that if E_1 & E_2 satisfy Caratheodory Criteria (1), then $E_1 \cap E_2$ will also satisfies (1).

From the bounded case $(n, n+1] \in \mathcal{M}$ iff $(n, n+1]$ satisfies (1). (Notice this.)

Hence, each of $E_n = E\cap(n, n+1]$ satisfies (1). Thus, by the bounded case, $E_n \in \mathcal{M}$.

Therefore, $E = \bigcup E_n \in \mathcal{M}$. ($\because \mathcal{M}$ is closed under countable union).

Lemma: E_1 & E_2 satisfy (1), then $E_1 \cap E_2$, $(E_1 \setminus E_2$, $E_1 \cup E_2$, $E_1^c)$ etc will satisfy (1). (58)

Proof: $m^*(A) = m^*(A \cap E_1) + m^*(A \setminus E_1)$ —(2)

$m^*(A) = m^*(A \cap E_2) + m^*(A \setminus E_2)$ —(3)

By replacing $A \mapsto A \cap E_1$, $A \setminus E_1$ in (3) and then using them in (2), implies,

$$m^*(A) \geq m^*(A \cap (E_1 \cap E_2)) + m^*(A \cap E_1 \setminus E_2)$$

$$+ m^*((A \setminus E_1) \cap E_2) + m^*(A \setminus E_1 \setminus E_2)$$

$$\geq m^*(A \cap (E_1 \cap E_2))$$

$$+ m^*[A \cap \{(E_1 \Delta E_2) \cup (E_1 \cup E_2)^c\}].$$

Since $E_1 \cup E_2 = (E_1 \Delta E_2) \cup (E_1 \cap E_2)$.

$$m^*(A) \geq m^*(A \cap (E_1 \cap E_2)) + m^*(A \cap (E_1 \cap E_2)^c)$$

Hence, $E_1 \cap E_2$ satisfies (1). As E satisfies (1) $\Rightarrow$ E^c do so,

$(E_1 \cup E_2)^c = E_1^c \cap E_2^c$ etc. will gives the other conclusions.

Corollary: If $\{E_i\}_{i=1}^{\infty}$ satisfies (1), then so is $\cup E_i$.

(Hint: let $E = \cup E_i$, then $E \in M \Rightarrow E$ satisfies (1), In fact, (I) is equivalent to say that

$$m^*(I) = m^*(I \cap E) + m^*(I \setminus E) \quad \text{---} (*)$$

holds for each bounded open interval $I \subset \mathbb{R}$.

First, let $m^*(A) = \infty$. Then $(*)$ is true for A instead of I. Now, consider $A = O$ & $m^*(O) < \infty$. Then $O = \bigcup_{n=1}^{\infty} I_n$ and $b_n - a_n < \infty$, $\forall\, n \geq 1$.

Hence, $O \in \mathcal{M}$ & satisfies $(*)$ in place of I.

Finally, let $m^*(A) < \infty$. Then, $\exists$ a G_δ-set $G \supset A$ s.t. $m^*(G) = m^*(A)$, where $G = \bigcap_{n=1}^{\infty} O_n$, O_n - open in $\mathbb{R}$. Since $(*)$ is true for each $O_n \Rightarrow G = \bigcap O_n \in \mathcal{M}$.

$\Rightarrow (*)$ holds for G. Thus,

$$m^*(A) = m^*(G) = m^*(G \cap E) + m^*(G \setminus E)$$
$$\geq m^*(A \cap E) + m^*(A \setminus E).$$

Notice that once again we come to a conclusion that measurability of E is equivalent to test measurability of each section of E by bounded intervals.

This will give way to generalize m^* to an ~~ab~~ abstract set rather real line.

Observation:

Set function $m^*: P(\mathbb{R}) \longrightarrow [0,\infty]$ s.t

(i) $m^*(\phi) = 0$ (definitness)

(ii) $m^*(A) \leq m^*(B)$, if $A \subseteq B \subseteq \mathbb{R}$. (monotone)

(iii) $m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$ (countable subadditivity)

Notice that (i) can be replaled by any set $E \in \mathcal{M}$ with $m(E) < \infty$. For this, we have

$$E = E \dot{\cup} \phi \Rightarrow m(E) = m(E) + m(\phi).$$
$$\Rightarrow m(\phi) = 0.$$

(ii) is followed by (iii). $B = (B \setminus A) \cup A$

$\Rightarrow m^*(B) = m^*(B \setminus A) + m^*(A)$, if $A, B \in \mathcal{M}$

i.e. if $m: \mathcal{M} \longrightarrow [0,\infty]$, $(\because m = m^*|_{\mathcal{M}})$

then (ii) can be merge with countable additivity.

Outer measures:

Let X be a non-empty set. A set function $\mu^*: P(X) \rightarrow [0,\infty]$ is said to be an outer measure on X if

(i) $\mu^*(\phi) = 0$ (definitness property).

(ii) $\mu^*(A) \leq \mu^*(B)$, if $A \subseteq B \subseteq X$.
(monotone property)

(iii) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$, for all sequence of sets $\{A_i\}$ in $P(X)$. (Countable subadditivity)

Ex. If $\mu^*(A) = \begin{cases} 0 & \text{if } A = \phi \\ 1 & \text{if } A \neq \phi \end{cases}$

is an outer measure on X.

Ex. $\mu^*(A) = \begin{cases} 0 & \text{if } A = \phi \\ \infty & \text{if } A \neq \phi \end{cases}$

is an outer measure on X.

Ex. Let X be an infinite set. Then ~~for~~

$\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is a finite set} \\ 1 & \text{if } A \text{ is not a finite set} \end{cases}$

does not define an outer measure on X, as μ^* fails to be countably subadditive. If $\{A_i\}_{i=1}^{\infty}$ is a disjoint family of finite non-empty subsets in X. Then

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 \not< 0 = \sum_{i=1}^{\infty} \mu^*(A_i).$$

This leads to seperate out those sets which satisfy

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu^*(A_i).$$