Def^n: A set $E \subset X$ is said to be μ^*-measurable (or m'sble) if (62)

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E),$$

$\forall A \subseteq X$.

By Countable subadditivity of μ^* we only need to verify

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E),$$

$\forall A \subseteq X$. —— (1)

Since (1) is symmetric in E & E^c, it follows that if E is m'sble then E^c will also m'sble.

Let M_{μ^*} denote the class of all μ^*-m'sble sets in X. Then $\phi, X \in M_{\mu^*}$.

If $\mu^*(E) = 0$, then for any $A \subseteq X$,

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E)$$
$$= 0 + \mu^*(A \setminus E).$$

Hence $E \in M_{\mu^*}$.

The space M_{μ^*} is a complete measure space, as if $\mu^*(E) = 0$ then $\mu^*(F) = 0$, $\forall F \subseteq E$. Hence $F \in M_{\mu^*}$

Let $\mu = \mu^*|_{M_{\mu^*}}$. Then we will show that μ is countably additive on M_{μ^*}.

Before proving this assertion, we need to show that $\mathcal{M}_{\mu^*}$ is closed under countable union. For this, let $E_1, E_2 \in \mathcal{M}_{\mu^*}$. Then $E_1 \cap E_2 \in \mathcal{M}_{\mu^*}$. The proof is same as done in lemma (p. 58) for m^*.

Thus, $E_1 \cup E_2 = (E_1^c \cap E_2^c)^c \in \mathcal{M}_{\mu^*}$.

By induction, it follows that if $\{E_i\}_{i=1}^n$ are msble, then $\bigcup_{i=1}^n E_i \in \mathcal{M}_{\mu^*}$.

<u>Lemma</u>: If $\{E_i\}_{i=1}^n$ is a disjoint family of sets in $\mathcal{M}_{\mu^*}$, then for any $A \subseteq X$,

$$\mu^*\left(A \cap \left(\bigcup_{i=1}^n E_i\right)\right) = \sum_{i=1}^n \mu^*(A \cap E_i)$$

<u>Proof</u>: Since $E_1 \in \mathcal{M}_{\mu^*}$,

$$(*) \quad \mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \setminus E_1),$$

$\forall A \subseteq X$. Replace $A \longmapsto A \cap (E_1 \cup E_2)$ in $(*)$. Then

$$\mu^*(A \cap (E_1 \cup E_2)) = \mu^*\{(A \cap (E_1 \cup E_2)) \cap E_1\} + \mu^*\{A \cap (E_1 \cup E_2) \setminus E_1\}$$

$$= \mu^*(A \cap E_1) + \mu^*(A \cap E_2),$$

(Since $E_1 \cap E_2 = \phi$).

Hence, by induction, the above lemma follows.

Further, if $\{E_i\}_{i=1}^{\infty}$ is disjoint family in M_{μ^*}, then by monotone property of μ^*, we get

$$\mu^*\left(A \cap \left(\overset{\infty}{\underset{i=1}{\dot{\cup}}} E_i\right)\right) \geq \mu^*\left(A \cap \left(\overset{n}{\underset{i=1}{\dot{\cup}}} E_i\right)\right)$$

(64)

$$= \sum_{i=1}^{n} \mu^*(A \cap E_i), \text{ (by previous lemma)}.$$

letting $n \to \infty$, we have

$$\mu^*\left(A \cap \left(\overset{\infty}{\underset{i=1}{\dot{\cup}}} E_i\right)\right) \geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i).$$

By countable subadditivity of μ^*, it follows that

$$\mu^*\left(A \cap \left(\overset{\infty}{\underset{i=1}{\dot{\cup}}} E_i\right)\right) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i), \ \forall A \subseteq X.$$

For $A = X$, we get

$$\mu\left(\overset{\infty}{\underset{i=1}{\dot{\cup}}} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Thus, μ is countably additive on M_{μ^*}.

<u>Cor</u>: If $\{E_i\}_{i=1}^{\infty}$ be a family in M_{μ^*}, then $\overset{\infty}{\underset{i=1}{\cup}} E_i \in M_{\mu^*}$.

Proof: Let $E = \overset{\infty}{\underset{i=1}{\cup}} E_i$. Define $E_i' = E_i \setminus \overset{i-1}{\underset{k=1}{\cup}} E_k$.

Then $E_i' \in M_{\mu^*}$ (by finite case),

$$E = \overset{\infty}{\underset{i=1}{\dot{\cup}}} E_i', \quad E_i' \cap E_j' = \phi, \text{ if } i \neq j.$$

Hence, w.l.g., we can assume that

$\{E_i\}_{i=1}^{\infty}$ is a disjoint family in M_{μ^*} and $E = \dot{\bigcup}_{i=1}^{\infty} E_i$. Let $E_n = \dot{\bigcup}_{i=1}^{n} E_i$. (65)

Then

$$\mu^*(A) = \mu^*(A \cap E_n) + \mu^*(A \cap E_n^c)$$
$$\geq \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^*(A \cap E^c), \ \forall n \geq 1.$$

(by previous lemma).

letting $n \to \infty$, we get

$$\mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap E^c)$$
$$= \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Hence $E = \dot{\bigcup}_{i=1}^{\infty} E_i \in M_{\mu^*}$ for any family $\{E_i\}_{i=1}^{\infty} \subset M_{\mu^*}$. Thus, M_{μ^*} is a σ-algebra.

Notice that

$$\mu : M_{\mu^*} \longrightarrow [0, \infty] \text{ s.t.}$$

(i) $\mu(\phi) = 0$,

(ii) $\mu\left(\dot{\bigcup}_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$.

Here, monotonicity of μ ~~be~~ will be followed by (ii). $\mu(A \dot{\cup} (B \setminus A)) = \mu(A) + \mu(B \setminus A)$, for $A, B \in M_{\mu^*}$ & $A \subset B$.

Thus, if S is any σ-algebra of sets in X, then a set function

$$\mu : S \longrightarrow [0, \infty]$$

satisfies (i) $\mu(\phi) = 0$, (ii) $\mu\left(\dot{\bigcup}_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$, is called a measure on (X, S).

Now, we will elaborate the idea of σ-algebra.

Def^n: Let $X \neq \phi$. Let $S \subset P(X)$ be such that

(i) $\phi \in S$

(ii) $A \in S \Rightarrow A^c \in S$

(iii) $\{A_i\}_{i=1}^{\infty} \subset S \Rightarrow \bigcup_{i=1}^{\infty} A_i \in S$.

Then S is called a σ-algebra on X and (X, S) is called measurable space with each member of S as measurable set.

Ex. $S_0 = \{\phi, X\}$ and $S_1 = P(X)$ are the smallest and the largest σ-algebra on X respectively.

Ex. For $X \neq \phi$, let $S = \{A \subseteq X : A \text{ or } A^c \text{ is countable}\}$.

Then S is a σ-algebra.

Here, $\phi, X \in S$, and if $A \in S \Rightarrow A^c \in S$.

Let $\{E_i\}_{i=1}^{\infty} \subset S$, and write

$I_1 = \{i \in \mathbb{N} : A_i \text{ countable}\}$ &

$I_2 = \{i \in \mathbb{N} : A_i^c \text{ countable}\}$.

Then $\left(\bigcup_{i \in \mathbb{N}} A_i\right)^c = \left\{\left(\bigcup_{i \in I_1} A_i\right) \cup \left(\bigcup_{i \in I_2} A_i\right)\right\}^c = \left(\bigcup_{i \in I_1} A_i\right)^c \cap \left(\bigcup_{i \in I_2} A_i\right)^c$

$=$ Countable.

Ex. Let X be an infinite set. Then

$$S = \{A \subseteq X : A \text{ or } A^c \text{ is finite}\}$$

is an algebra (i.e. closed under complement, and finite union, $\phi \in S$), but S is not a σ-algebra.
(Hint: First do for $X = \mathbb{N}$, $A_n = \{2n\}$.)
(Write $X = X_1 \cup X_2$, X_1 - countable & X_2 infinite.)

Ex. Let $\mathcal{A}$ be an algebra of sets in X. Then show that $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions.
(i.e. $A_i \uparrow$, $A_i \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$).
(Hint: $A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$, $B_i = A_1 \cup \cdots \cup A_i$.)

<u>σ-algebra generated by a family of sets.</u>

Let $X \neq \phi$, $A \subset X$, $\mathcal{E} = \{A\} \subset P(X)$.
Then $\{\phi, X, A, A^c\}$ is a σ-algebra, & we write $\sigma(\mathcal{E}) = \{\phi, X, A, A^c\}$, the σ-algebra generated by $\mathcal{E}$. However, if $\mathcal{E}$ is a large family of sets, it is difficult to immumerate the sigma algebra generated by $\mathcal{E}$. However, ~~So~~ $\mathcal{E}$ is contained in many σ-algebras, like $P(X)$, the σ-algebra generated by $\mathcal{E}$ will be $\sigma(\mathcal{E}) = \bigcap\{S : \mathcal{E} \subset S, S \text{ } \sigma\text{-algebra}\}$.

Hence, $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$. Let $\mathcal{B}(\mathbb{R})$ be the σ-algebra generated by all open sets. Then $\mathcal{B}(\mathbb{R})$ can be generated by any of the following collections. (68)

(i) F_1 = all closed sets in $\mathbb{R}$

(ii) $F_2 = \{(-\infty, b] : b \in \mathbb{R}\}$

(iii) $F_3 = \{(a, b] : a<b,\ a, b \in \mathbb{R}\}$.

Let $\mathcal{B}_i = \sigma(F_i)$; $i = 1, 2, 3$. We prove that $\mathcal{B}(\mathbb{R}) \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_3 \supseteq \mathcal{B}(\mathbb{R})$.

Since $\mathcal{B}(\mathbb{R})$ contains all open sets & closed under complement, $\mathcal{B}(\mathbb{R}) \supset F_1$. Given $\mathcal{B}(\mathbb{R})$ is a σ-algebra, $\mathcal{B}(\mathbb{R}) \supseteq \sigma(F_1)$.

As $(-\infty, b]$ is closed, it follows that

$$F_1 \supset F_2 \Rightarrow \sigma(F_1) \supseteq \sigma(F_2) \quad (\text{Ex.})$$

$$\Rightarrow \mathcal{B}_1 \supseteq \mathcal{B}_2.$$

Next, $(a, b] = (-\infty, b] \cap ((-\infty, a])^c$, we get

$$\sigma(F_2) \supset F_3 \Rightarrow \mathcal{B}_2 \supseteq \mathcal{B}_3.$$

Notice that $(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] \subset \mathcal{B}_3 = \sigma(F_3)$.

Hence, each ~~bounded~~ open set of the form $O = \bigcup_{n=1}^{\infty} (a_j, b_j) \in \sigma(F_3)$.

Since $(-\infty, b] = (-\infty, a] \cup (a, b) \in \sigma(F_3)$, and similarly, $(a, \infty) \in \sigma(F_3)$. It follows that B_3 contains each open subset of $\mathbb{R}$. Thus,

$$B_3 \supseteq \sigma\{O \subseteq \mathbb{R} : O \text{ open}\} = B(\mathbb{R}).$$ 69

If μ is a measure on the σ-algebra S of subsets of X, the (X, S, μ) is called measure space.

* (X, S, μ) is called finite measure space if $\mu(X) < \infty$.
* (X, S, μ) is called σ-finite measure space if X can be expressed as countable union of sets of finite measure. (i.e. $X = \bigcup_{i=1}^{\infty} E_i$, $\mu(E_i) < \infty$, $\forall i$).

Ex. $(\mathbb{R}, M, m)$ is a σ-finite measure space but not finite measure space.

Ex. Let $Y \subset X$ & S be a σ-algebra on X. Then $S \cap Y = \{A \cap Y : A \in S\}$ is a σ-algebra which can be thought as relative σ-alg. of Y.

Ex. $([0,1], M/_{[0,1]}, m/_{[0,1]})$ is a finite measure space.

Ex. $(\mathbb{R}, P(\mathbb{R}), \mu)$ with $\mu(A) = \begin{cases} \#(A) & \text{if } A \text{ finite} \\ \infty & \text{otherwise} \end{cases}$

then μ is neither finite nor σ-finite.

Proposition: Let (X, S, μ) be a σ-finite measure space. Then

(i) $\exists$ an $\uparrow$ seq^n in S that satisfies σ-finite condition.

(ii) $\exists$ a disjoint seq^n in S that satisfies σ-finite condition.

Proof: Given that $X = \bigcup_{i=1}^{\infty} E_i$, $\mu(E_i) < \infty$.

(i) Let $F_n = \bigcup_{i=1}^{n} E_i$. Then $F_n \uparrow$ & $\mu(F_n) < \infty$, and $X = \bigcup_{n=1}^{\infty} F_n$.

(ii) Let $G_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$, Then $X = \bigcup G_n$, $\mu(G_n) < \infty$ and $G_n \cap G_m = \phi$, if $n \neq m$.

Ex. Let (X, S, μ) be a measure space.

(i) If $E, F \in S$, then for $\mu(F) < \infty$ and $F \subset E$, we have
$$\mu(E \setminus F) = \mu(E) - \mu(F).$$

(ii) For any $E, F \in S$,
$$\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \cup F).$$

Proof (i) is trivial as $\mu((E \setminus F) \cup F) = \mu(E \setminus F) + \mu(F)$.

(ii) $E = (E \setminus F) \cup (E \cap F)$

$F = (F \setminus E) \cup (E \cap F)$

$\therefore \mu(E) + \mu(F) = \mu(E \setminus F) + \mu(F \setminus E) + \mu(E \cap F) + \mu(E \cap F)$

$= \mu(E \cup F) + \mu(E \cap F)$.

Proposition:

Let (X, S, μ) be a measure space.

(i) If $\{E_n\}_{n=1}^{\infty}$ is an $\uparrow$ seqn in S, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

(ii) If $\{E_n\}_{n=1}^{\infty}$ is a $\downarrow$ sequence in S, with $\mu(E_1) < \infty$, then $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$.

(Hint: proof is similar to that of Lebesgue measure on.)

Pre-measures:

Let A be an algebra of sets in X.

A set function $\mu_0 : A \longrightarrow [0, \infty]$

satisfying (i) $\mu_0(\phi) = 0$ &

(ii) If $\{E_n\}_{n=1}^{\infty}$ is a disjoint sequence in A with $\dot{\bigcup} A_n \in A$, then

$$\mu_0\left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) = \sum_{n=1}^{\infty} \mu_0(A_n)$$

is called a premeasure on A. Obviously, μ_0 is finitely additive.

Now, for $A \subseteq X$, define

$$\mu^*(A) = \inf\left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : A \subseteq \bigcup E_i, E_i \in A \ \forall E_i \in A \right\}$$

Then μ^* is an outer-measure on X.

Proof: (i) $\mu^*(\phi) = 0$ & (ii) $\mu^*(A) \le \mu^*(B)$, if $A \subseteq B$ are obvious. For countably subadditive, let $\{A_n\}_{n=1}^{\infty} \subset P(X)$. Then for each $\epsilon > 0$, $\exists$ a cover $\{E_{n,j}\}$ of A_n such that

$$\sum_{j=1}^{\infty} \mu_0(E_{n,j}) < \mu^*(A_n) + \epsilon/2^n.$$

Hence, $\{E_{n,j} : n \in N, j \in N\}$ is a cover of $\bigcup_{n=1}^{\infty} A_n$. Thus,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \mu_0(E_{n,j}) \le \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon,$$

$\forall \epsilon > 0$. Hence,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

Lemma: Any set $E \in \mathcal{A}$ is a μ^*-measurable set and $\mu^*(E) = \mu_0(E)$.

Proof: Let $A \subseteq X$, and $\epsilon > 0$. Then $\exists$ a cover $\{E_i\}_{i=1}^{\infty}$ of A s.t.

$$\epsilon + \mu^*(A) > \sum_{i=1}^{\infty} \mu_0(E_i) \quad \text{——— (1)}$$

Now, $E_j = (E_j \cap E) \dot\cup (E_j \cap E^c)$.

From (1), $\epsilon + \mu^*(A) > \sum_{j=1}^{\infty} \{\mu_0(E_j \cap E) + \mu_0(E_j \cap E^c)\}$ ($\because \mu_0$ is finitely additive).

$$\epsilon + \mu^*(A) > \mu^*\left(\left(\bigcup_{i=1}^{\infty} E_i\right) \cap E\right) + \mu^*\left(\left(\bigcup_{i=1}^{\infty} E_i\right) \cap E^c\right)$$

$$\geq \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

$\forall\, \epsilon > 0$, hence

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E).$$

That is $E \in \mathcal{M}_{\mu^*}$ (class of μ^*-measurable sets).

Next, $\mu^*(E) \leq \mu_0(E)$, since E covers itself.

On the other hand, let $E \subset \bigcup_{j=1}^{\infty} E_j$, $E_j \in \mathcal{A}$.

Write $E_j' = \left(E_j \setminus \bigcup_{i=1}^{j-1} E_i\right) \cap E$. Then $E_j' \cap E_k' = \phi$, if $j \neq k$, & $E = \bigcup_{j=1}^{\infty} E_j'$, $E_j' \in \mathcal{A}$.

Now, $\mu_0(E) = \mu_0\left(\bigcup_{j=1}^{\infty} E_j'\right) = \sum_{j=1}^{\infty} \mu_0(E_j') \leq \sum_{j=1}^{\infty} \mu_0(E_j)$.

$\Rightarrow \mu_0(E) \leq \mu^*(E)$. Hence $\mu_0(E) = \mu^*(E)$.

Further, let $\mu = \mu^*/\mathcal{M}_{\mu^*}$. Then, as usual, μ is a measure on $\mathcal{M}_{\mu^*}$. Notice that μ extends μ_0 to $\mathcal{M}_{\mu^*}$.

<u>Theorem</u>: If μ_0 is a σ-finite premeasure on $\mathcal{A}$. Then $\exists !$ measure μ on $\mathcal{M}_{\mu^*}$ s.t. $\mu/\mathcal{A} = \mu_0$. (i.e $\exists!$ μ on $\mathcal{M}_{\mu^*}$ that extends μ_0).

<u>Proof</u>: Let ν be another extension of μ_0. That is, $\nu/\mathcal{A} = \mu_0$. Then for $E \in \mathcal{M}_{\mu^*}$,

and a cover $\{E_i\}_{i=1}^{\infty}$ of E. we have

$$\nu(E) \le \sum_{i=1}^{\infty} \nu(E_i) \le \sum_{i=1}^{\infty} \mu_0(E_i) \quad (\because E_i \in \mathcal{A}).$$

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$$\Rightarrow \nu(E) \le \mu^*(E) = \mu(E). \quad \text{———— (1)}$$

If $\mu(E) < \infty$, then for $\epsilon > 0$, $\exists$ a set

$$F = \bigcup_{i=1}^{\infty} E_i \supset E \text{ s.t.}$$

$$\mu(F) \le \sum \mu_0(E_i) < \mu(E) + \epsilon.$$

$$\Rightarrow \mu(F \setminus E) < \epsilon \quad (\because \mu(E) < \infty).$$

Now, $\nu(F) = \lim_{n\to\infty} \nu\left(\bigcup_{i=1}^{n} E_i\right) = \lim \mu\left(\bigcup_{i=1}^{n} E_i\right) = \mu(F).$

Since $E \subset F = \bigcup_{i=1}^{\infty} E_i$, we get

$$\begin{aligned} \mu(E) \le \mu(F) &= \nu(F) \quad (\because F = \cup E_i \in \mathcal{A}) \\ &= \nu(E) + \nu(F \setminus E) \\ &\le \nu(E) + \mu(F \setminus E) \quad (\text{by (1), as (1) holds for all } E \in \mathcal{M}_{\mu^*}) \\ &< \nu(E) + \epsilon, \ \forall \epsilon > 0. \end{aligned}$$

$$\Rightarrow \mu(E) \le \nu(E) \le \mu(E).$$

$\Rightarrow \mu(E) = \nu(E), \ \forall \ E \in \mathcal{M}_{\mu^*}$ with $\mu(E) < \infty$. If $\mu(E) = \infty$, then by the fact that μ_0 is σ-finite, we have

$$X = \bigcup_{i=1}^{\infty} E_i, \quad \mu_0(E_j) < \infty, \ E_j \in \mathcal{A}.$$

$$\begin{aligned} \therefore \mu(E) = \mu\left(\bigcup_{i=1}^{\infty} (E \cap E_i)\right) &= \sum \mu(E \cap E_i) \\ &= \sum \nu(E \cap E_i) \\ &= \nu(E). \ (\text{by finite case}) \end{aligned}$$