Measurable functions:

Let $\mathcal{J}_u$ = Collection of all open subsets of $\mathbb{R}$ w.r.t usual metric u on $\mathbb{R}$.

$$= \left\{ O \subseteq \mathbb{R} : O = \bigcup_{n=1}^{\infty} I_n,\ I_n = (a_n, b_n) \right\},$$

and $\mathcal{M}$ = class of all L-msble subsets of $\mathbb{R}$.

$\mathcal{J}_{d_0}$ = Collection of all open subsets of $\mathbb{R}$ w.r.t. d_0 - the discrete metric on $\mathbb{R}$ = $P(\mathbb{R})$.

$\Rightarrow \mathcal{J}_u \subsetneq \mathcal{M} \subsetneq \mathcal{J}_{d_0} = P(\mathbb{R})$.

Since $\mathcal{J}_u$ is not closed under countable intersection (& complement) of open sets,

$\Rightarrow \mathcal{J}_u \subsetneq \mathcal{M}$, $\mathcal{M} \subsetneq \mathcal{J}_{d_0}$, because every subset of $\mathbb{R}$ need not be L-msble.

Consider $f : (\mathbb{R}, \mathcal{J}_u) \xrightarrow{cont} (\mathbb{R}, \mathcal{J}_u)$. Then

$f^{-1}(O) \in \mathcal{J}_u$, $\forall\, O \in \mathcal{J}_u$ (from range)

Now, if $f : (\mathbb{R}, \mathcal{M}) \longrightarrow (\mathbb{R}, \mathcal{J}_u)$, what we can say about $f^{-1}(O)$?

If f is continuous on $(\mathbb{R}, \mathcal{J}_u)$, then $f^{-1}(O) \in \mathcal{M}$, because $f^{-1}(O)$ is open.

In addition, Consider $f(x) = \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$. Then f cannot be made continuous at '0', but $f(x) = \infty$ iff $x = 0$ (important!)

If we want to take $f(x) = \frac{1}{x}$ into consideration, we have to extend range $(-\infty, \infty)$ to $[-\infty, \infty]$. (76)

Let $\mathbb{R} = (-\infty, \infty)$ and $\overline{\mathbb{R}} = [-\infty, \infty]$.

Therefore, the sets $[-\infty, a)$ & $(b, \infty]$ for $a, b \in \mathbb{R}$, should be added to $\mathcal{J}_u$.

That is, $\overline{\mathcal{J}_u} = \mathcal{J}_u \cup \{[-\infty, a), (b, \infty], a, b \in \mathbb{R}\}$.

Notice that $[-\infty, a) \cup [b, \infty]$ is the complement of $[a, b]$ in $\mathbb{R}$ unions with $\{\pm\infty\}$. That is, $(\overline{\mathbb{R}}, \overline{\mathcal{J}_u})$ is two point compactification of $(\mathbb{R}, \mathcal{J}_u)$. But $f(x) = \frac{1}{x}$ is still not continuous. because, for $a > 0$,

$$f^{-1}\{[\infty, a)\} = (-\tfrac{1}{a}, 0] \notin \mathcal{J}_u.$$

Def^n: $f : (\mathbb{R}, M) \longrightarrow (\overline{\mathbb{R}}, \overline{\mathcal{J}_u})$ is said to be Lebesgue measurable if

$$f^{-1}(O) \in M, \quad \forall\ O \in \overline{\mathcal{J}_u}.$$

Hence $f(x) = \frac{1}{x}$, $x \neq 0$, is L-m'ble. (77)

Since $O \in \overline{J}_4$ can be expressed as countable union & finite intersection of the form $[-\infty, a)$ & $(b, \infty]$, etc, it is enough to consider $O = (b, \infty]$ or $[-\infty, a)$.

Thus, $f : (\mathbb{R}, M) \longrightarrow (\overline{\mathbb{R}}, \overline{J}_4)$ or $\overline{\mathbb{R}}$ is L-m'ble if $f^{-1}\{(\alpha, \infty]\} \in M$, $\forall \alpha \in \mathbb{R}$.

Similarly, if $f : (X, S) \longrightarrow \overline{\mathbb{R}}$, then f is said to be S-m'ble if $f^{-1}\{(\alpha, \infty]\}$ belongs to S, $\forall \alpha \in \mathbb{R}$.

Lemma: If $f : (X, S) \to \overline{\mathbb{R}}$. Then FAE

(i) $f^{-1}\{(\alpha, \infty]\} \in S$, $\forall \alpha \in \mathbb{R}$,

(ii) $f^{-1}\{[\alpha, \infty]\} \in S$, $\forall \alpha \in \mathbb{R}$,

(iii) $f^{-1}\{[-\infty, \alpha)\} \in S$, $\forall \alpha \in \mathbb{R}$

(iv) $f^{-1}\{[-\infty, \alpha]\} \in S$, $\forall \alpha \in \mathbb{R}$

(v) $f^{-1}\{(a, b)\} \in S$, $\forall a, b \in \mathbb{R}$ and $f^{-1}\{\pm\infty\} \in S$.

Proof: (i) $\Rightarrow$ (ii):

$$[\alpha, \infty] = \bigcap_{n=1}^{\infty} (\alpha - \tfrac{1}{n}, \infty].$$

Let $x \notin$ RHS, then $\exists\, n_0 \in \mathbb{N}$ s.t.

$$x \le \alpha - \frac{1}{n_0} < \alpha. \text{ Hence } x \notin \text{LHS}.$$

Since S is closed under complement, (ii) $\Rightarrow$ (iii). Now, (iii) $\Rightarrow$ (iv) because

$$[-\infty, \alpha] = \bigcap_{n=1}^{\infty} [-\infty, \alpha + \tfrac{1}{n}).$$

Now, (iv) $\Rightarrow$ (i). Thus, (i) to (iv) are equivalent.

Here $f^{-1}\{\infty\} = \bigcap f^{-1}\{(n,\infty]\} \in S$, by (i).

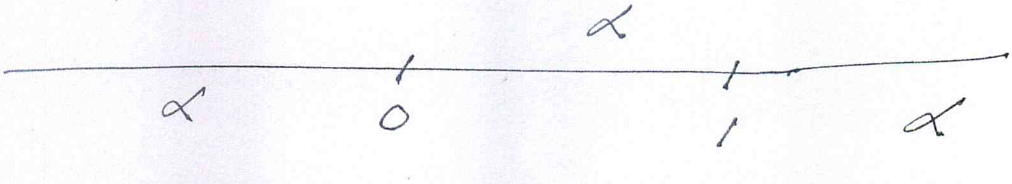
& $f^{-1}\{-\infty\} = \bigcap f^{-1}\{[-\infty, -n)\} \in S$, by (iii).

Also, $(a,b) = (a,\infty] \cap [-\infty, b)$, we get (v). Finally, (v) $\Rightarrow$ (i), is followed by

$$(a,\infty] = (a,\infty) \cup \{\infty\} = \bigcup_{n \geq |a|} (a,n) \cup \{\infty\}.$$

Ex. Let $E \in S$, and define

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$



$$f^{-1}\{(\alpha,\infty]\} = \{x \in X : f(x) > \alpha\} = \begin{cases} \mathbb{R}, & \alpha < 0 \\ E & 0 \le \alpha < 1 \\ \phi & \alpha \ge 1 \end{cases}$$

Hence the characteristic χ_E is measurable iff $E \in S$.

Ex. Let (X,S) be a measurable space. Then const function is S-m'ble.

Let $f(x) = c$, $\forall x \in X$. If c is finite

$$\{x \in X : f(x) > \alpha\} = \begin{cases} \phi & \alpha < c \\ X & \alpha \geq c. \end{cases}$$

If $f(x) = \infty$, $\forall x \in X$, then

$$\{x \in X : f(x) > \alpha\} = X.$$

Notice that for $\alpha \in \mathbb{R}$, $\exists\ r_n \in \mathbb{Q}$ s.t $r_n \uparrow \alpha$. Thus $f(x) \geq \alpha \Rightarrow f(x) > r_n$, $\forall n$.

i.e, $\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > r_n\}$.

Thus f is S-m'ble iff $f^{-1}\{(r, \infty]\} \in S$, for all $r \in \mathbb{Q}$.

Ex. If D is a dense set in $\mathbb{R}$, then f is S-m'ble iff $f^{-1}\{(r, \infty]\} \in S$, $\forall r \in D$

Let $\alpha_1 \in (\alpha - 1, \alpha]$, $\alpha_1 \in D$ and construct $\alpha_2 \in (\alpha - \frac{1}{2}, \alpha] \cap (\alpha_1 - \frac{1}{2}, \alpha]$. Then by induction, $\alpha_n \uparrow$ & $\alpha - \frac{1}{n} < \alpha_n \leq \alpha$, $\forall n \geq 1$

Hence, the conclusion of the exercise is followed.

Ex. If $f, g : (X,S) \longrightarrow \overline{\mathbb{R}}$ are m'ble and $f(x) + g(x) \neq \infty - \infty$, for any $x \in X$. Then $f+g$ is m'ble.

For this, we need to show that

$$A = \{x \in X : f(x) + g(x) = \pm\infty\} \in S$$

& $B = \{x \in X : \infty > f(x) + g(x) > \alpha\} \in S, \ \forall \alpha \in \mathbb{R}$.

Now, $A = \{x \in X : f(x) = \pm\infty\}$ if $g(x)$ is finite (or otherwise). Thus, $A \in S$.

For $x \in B$, $\alpha < f(x) + g(x) < \infty$. Then $\exists\, r_x \in \mathbb{Q}$ such that

$$f(x) > r_x > \alpha - g(x).$$

$$\Rightarrow x \in \bigcup_{r \in \mathbb{Q}} \left\{\{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}\right\}$$

Hence

$$B = \bigcup_{r \in \mathbb{Q}} \left\{\{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}\right\} \in S.$$

Ex. If $f : (X, S) \longrightarrow \overline{\mathbb{R}}$ is m'ble, then

$$\{x : f^2(x) > \alpha\} = \begin{cases} \{x : f(x) > \sqrt{\alpha}\} \cup \{x : -f(x) > \sqrt{\alpha}\}, & \alpha \geq 0 \\ X & \text{if } \alpha < 0. \end{cases}$$

Hence f^2 is m'ble.

Ex. $fg = \frac{1}{4}\{(f+g)^2 - (f-g)^2\}$, implies that if f, g are m'ble then fg is m'ble.

Def^n: A property P is said to hold "almost everywhere", if the places (pts) where it fails has measure zero.

let (X, S, μ) be a measure space. Then

$$\mu^*\{x \in X : P \text{ is false}\} = 0.$$

Ex. let $f : (X, S, \mu) \longrightarrow \overline{\mathbb{R}}$ be such that $f = 0$ almost everywhere (a.e.), then f is msble, if (X, S, μ) is complete.

let $E = \{x \in X : f(x) \neq 0\}$. Then $\mu^*(E) = 0$.

$$\therefore \{x \in X : f(x) > \alpha\} = \begin{cases} E^c \cup A, \ A \subset E & \alpha < 0 \\ B \ , \ B \subset E & \alpha \geq 0. \end{cases}$$

Since (X, S, μ) is complete and $A, B \subset E$, implies $A, B \in S$. Thus, f is msble.

Notice that for $A \subset X$, we have

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \mu(E_i) : E_i \in S, \ A \subset \cup E_i\right\}$$
$$= \inf\{\mu(E) : E \in S, \ A \subset E\}.$$

Ex. If $f : (X, S) \longrightarrow \overline{\mathbb{R}}$ is msble. Then $|f|$ is also msble.

$$\{x : |f(x)| > \alpha\} = \begin{cases} \{x : f(x) > \alpha\} \cup \{x : f(x) < -\alpha\}, & \alpha \geq 0 \\ X & \alpha < 0. \end{cases}$$

But converse need not be true.

Ex. let $N \subset \mathbb{R}$ be non-measurable set w.r.t. m^*. Then

$$f(x) = \begin{cases} 1 & x \in N \\ -1 & x \notin N \end{cases}$$

is not L-integrable, but $|f| = 1$ is integrable.

Let $L(X,S)$ and $L(X,S,\mu)$ denote the space of all integrable functions on X.

Define $f^+ = \max\{f, 0\}$ & $f^- = -\min\{f, 0\}$.

Then $f^+ = \frac{f + |f|}{2}$ and $f^- = \frac{|f| - f}{2}$.

Hence, if $f \in L(X,S)$, then $f^+, f^- \in L(X,S)$.

Note that $f = f^+ - f^-$ & $|f| = f^+ + f^-$.

Ex. $f : [0, 2\pi] \to \mathbb{R}$, $f(x) = \sin x$, then

$$f^+(x) = \begin{cases} \sin x & \text{if } 0 \leq x \leq \pi \\ 0 & \text{if } \pi \leq x \leq 2\pi \end{cases}$$ and

$$f^-(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \pi \\ -\sin x & \pi \leq x \leq 2\pi \end{cases}$$

Ex. Let $f_n \in L(X,S)$. Then $\inf f_n$, $\sup f_n$, $\overline{\lim} f_n$, $\underline{\lim} f_n$ and $\lim f_n$ (if exists) are in $L(X,S)$.

$$\{x : \inf f_n(x) < \alpha\} = \bigcup \{x : f_n(x) < \alpha\}.$$

If $x \in$ LHS, then $\exists$ $n_0 \in N$ s.t. $f_{n_0}(x) < \alpha$.

Hence, $x \in$ RHS and vice-versa.

Lemma: Let $f : (X, S) \longrightarrow \mathbb{R}$. Then FAE:

(83)

(i) $f \in L(X, S)$

(ii) $f^{-1}(O) \in S$, $\forall$ open set $O \subseteq \mathbb{R}$.

(iii) $f^{-1}(F) \in S$, $\forall$ closed set $F \subseteq \mathbb{R}$.

(iv) $f^{-1}(B) \in S$, $\forall$ Borel set $B \subset \mathbb{R}$.

Proof: Since $f : X \longrightarrow \mathbb{R} = (-\infty, \infty)$, $f^{-1}\{\pm\infty\} = \phi$.

$f^{-1}(O) = \dot{\cup} f^{-1}(I_n) \in S$, where $O = \dot{\cup} I_n$.

$\therefore$ (i) $\Rightarrow$ (ii). Since S is σ-algebra, for F is a closed set,

$$f^{-1}(F) = f^{-1}(\mathbb{R} \setminus F^c) = f^{-1}(\mathbb{R}) - f^{-1}(F^c) \in S$$

$\Rightarrow$ (iii). Now, suppose

$$\mathcal{A} = \{ F \subset X : f^{-1}(F) \in S \}.$$

Then $\mathcal{A}$ is a σ-algebra containing $\mathcal{B}(\mathbb{R})$, hence (iii) $\Rightarrow$ (iv). Finally, (iv) $\Rightarrow$ (i) is obvious.

Let $L(\mathbb{R}, \mathcal{M}, m)$ be the space of all Lebesgue m'ble function and $L(\mathbb{R}, \mathcal{B}, m)$ be the space of all Borel m'ble functions. Then $L(\mathbb{R}, \mathcal{B}, m) \subsetneq L(\mathbb{R}, \mathcal{M}, m)$, since $\mathcal{B} \subsetneq \mathcal{M}$, because

$$\#(\mathcal{B}) = 2^{\aleph} = c, \quad \#(P(C)) = 2^{\#(C)} = 2^c.$$

Monotone functions:

Result: Let $f:(a,b) \longrightarrow \mathbb{R}$ be a monotone function. Then for $c \in (a,b)$, $f(c-)$ and $f(c+)$ both exist.

Proof: Let f be $\uparrow$. Then

$$f(c-) = \sup_{a<x<c} f(x) = L \leq f(c)$$

$$\& \quad f(c+) = \inf_{c<x<b} f(x) = M \geq f(c).$$

For $\epsilon > 0$, $\exists$ $x_0 \in (a,c)$ s.t. $f(x_0) > L - \epsilon$.
Let $\delta = c - x_0$, then for $x \in (c-\delta, c)$,

$$L + \epsilon > f(x) \geq f(x_0) > L - \epsilon$$

i.e. $x \in (c-\delta, c) \Rightarrow |f(x) - L| < \epsilon$.

Hence $f(c-) = \sup_{a<x<c} f(x) = L$.

Notice, that for $c<d$, $c,d \in (a,b)$

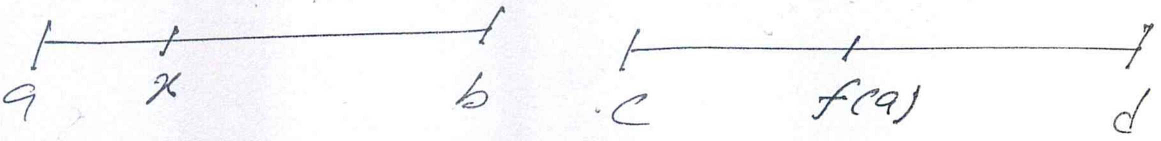
$f(c+) \leq f(d-)$. Hence,

$(f(c-), f(c+))$ and $(f(d-), f(d+))$ either both coincide or disjoint and can be imbedded into set of rationals $\mathbb{Q}$.

Hence, set of discontinuities of a monotone function is at most countable.

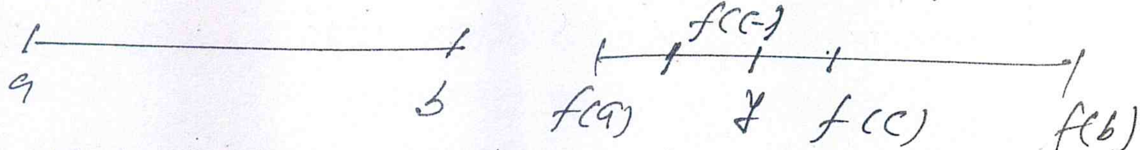
Ex. If $f:[a,b] \longrightarrow [c,d]$ is monotone and onto, then f is continuous.

Let f be $\uparrow$. Then $f(a)=c$ & $f(b)=d$.



If $f(a)>c$, then for $y \in [c, f(a)]$, $\nexists$ any $x \in [a,b]$ s.t. $f(x)=y$. If so, then $f(x)=y<f(a) \Rightarrow x<a$ ($\because f$ is $\uparrow$).

Further, if possible, let $f(c-)<f(c)$.



Then $y \in (f(c-), f(c))$ has no pre-image. If $\exists\ x_0 \in (a,c)$ st $f(x_0)=y$. Then

$L=\sup_{a<x<c} f(x) < f(x_0) < f(c)$, which contradict the fact that L is supremum on (a,c).

Thus, $f(c-)=f(c)=f(c+)$.

Ex. If $f:(a,b) \xrightarrow{\text{onto}} (c,d)$ is monotone, then f is continuous.

Proof is similar to the above case.

Observe that if f monotone onto, then f need not be one-one. However, if f is strictly monotone and onto, then $f^{-1} : (c,d) \to (a,b)$ is continuous, because in this case, f^{-1} is also strictly monotone. For this, (86)

if $f \uparrow_s$, then $y_1 < y_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)$.

If not, then $f^{-1}(y_1) \geq f^{-1}(y_2) \Rightarrow x_1 \geq x_2$

but $y_1 = f(x_1) < f(x_2) = y_2$.

Note that $f : [c,d] \xrightarrow{\text{onto}} (e,f)$ need not be continuous, if f is monotone, else $f([a,b])$ would be compact.

Finally, if $f : \mathbb{R} \to \mathbb{R}$ is one-one onto & cont, then f & f^{-1} both are continuous.

Ex. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be monotone function, then $E_\alpha = \{x \in I : f(x) > \alpha\} = I'$ & ϕ.

If $x' \in E_\alpha$, then $x' < x \leq b \Rightarrow f(x) > f(x') > \alpha$

$\Rightarrow [x', b] \subset E_\alpha$.

Let $x_0 = \inf\{x \in I : f(x) > \alpha\} = \inf E_\alpha$.

(i) If $x_0 = a$, then for $x \in I$, $\exists$ $x_1 \in E_\alpha$

s.t. $x_1 \leq x$ and $f(x) \geq f(x_1) > \alpha \Rightarrow x \in E_\alpha$.

$\Rightarrow I = E_\alpha$.

(ii) $a < x_0 \le b$. Then for $x > x_0$, $\exists\, x_1 \in E_\alpha$ such that $x_0 < x_1 < x$ and $f(x) > f(x_1) > \alpha$.

$\Rightarrow (x_0, b] \subset E_\alpha$.

If $x < x_0$, then $f(x) \le \alpha \Rightarrow x \notin E_\alpha$.

$\Rightarrow (x_0, b] \subseteq E_\alpha \subseteq [x_0, b]$.

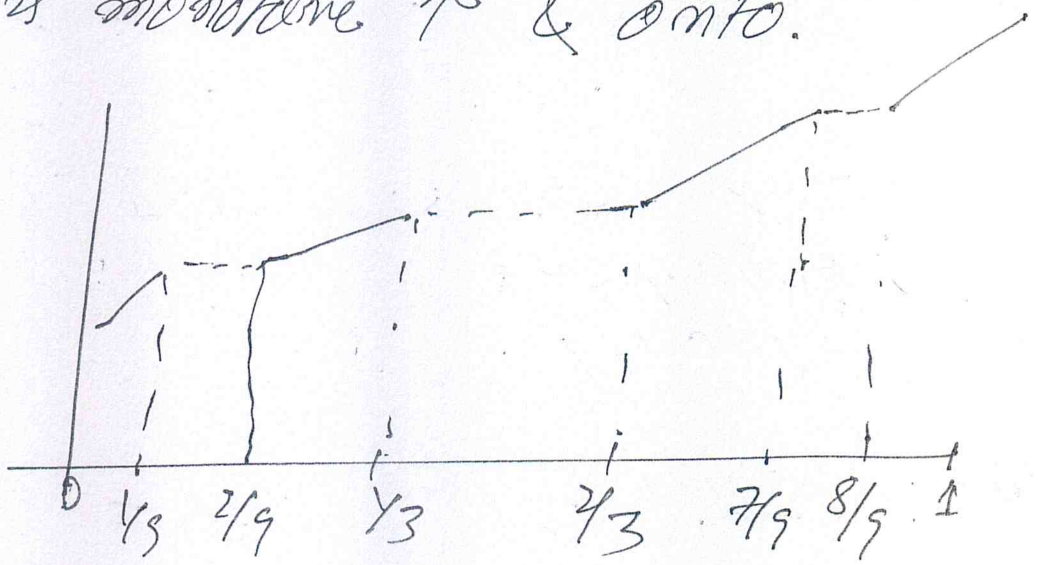
Thus, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is monotone, then f is Borel m'ble.

<u>non-Borel measurable sets.</u>

Let $f : C \longrightarrow [0,1]$ be defined by

$$f(x) = f\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) = \sum \frac{a_i}{2} \frac{1}{2^i}$$

Then f is monotone $\uparrow$ & onto.



We know that

$$C = [0,1] \setminus \left\{ \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \cup \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right) \cdots \right\}$$

Let $C = [0,1] \setminus \bigcup_{n=1}^{\infty} I_n$, $I_n = (a_n, b_n)$. (89)

Define $\tilde{f} : [0,1] \longrightarrow [0,1]$ by

$$\tilde{f}(C) = f(C) \quad \& \quad \tilde{f}(I_n) = \{a_n\}.$$

Then $\tilde{f}$ is monotone on $[0,1]$ and onto $[0,1]$. Hence $\tilde{f}$ is continuous.

Define $g : [0,1] \longrightarrow [0,2]$ by

$$g(x) = \tilde{f}(x) + x.$$

Then g is strictly increasing and onto function, Here $g(0) = 0$ &

$$g(1) = f(1) + 1 = f\left(\sum \frac{2}{3^i}\right) + 1 = 2.$$

For $0 < y < 2$, by IVT, $\exists$ $x \in (0,1)$ s.t $y = f(x)$.

Hence g is 1-1, onto cont.

claim $m(g(C)) = 1$.

$$m\{g([0,1] \setminus C)\} = m\{g(\cup I_n)\}$$
$$= m\{\cup g(I_n)\}$$
$$= \sum m\{g(I_n)\}$$
$$= \sum m\{a_n + I_n\} = 1.$$

ie $2 - m(g(C)) = 1 \Rightarrow m\{g(C)\} = 1.$

Hence $\exists\, B \in g(C)$ s.t $B \notin M$. Let $A = g^{-1}(B)$. Then $A \notin \mathcal{B}$. If $A \in \mathcal{B}$, then the fact that $h = g^{-1}$ is continuous, $h^{-1}(A) \in M$

$$\Rightarrow B = g(A) \in M, \text{ a contradiction.}$$

Note that $B = g(A) \subset g(C) \Rightarrow A \subset C$.

$$\Rightarrow A \subset C \in \mathcal{B}, \quad m(C) = 0,$$

but $A \notin \mathcal{B}$.

That is, $(\mathbb{R}, \mathcal{B})$ is not a complete measure space w.r.t m.

Simple functions:

Let $E_i \in S$, and $\alpha_i \in \overline{\mathbb{R}}$. Then

$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ is called a simple function on (X, S).

Hence φ is m'ble iff $\varphi(x) \neq \infty - \infty$ for any $x \in X$ and each of $E_i \in S$.

Notice that $\chi_{E_1} \chi_{E_2} = \chi_{E_1 \cap E_2}$ and

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2},$$

hence w.l.g. we can assume all of E_i's are pair-wise disjoint. (91)

Thus, the canonical representation of a simple function is

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}, \quad E_i \cap E_j = \phi \quad i \neq j$$
$$\alpha_i \in \overline{\mathbb{R}}.$$

<u>Simple functions are dense in $L(X,S)$.</u>

why we need the denseness of simple functions?

let f be R-integrable on $[0,1]$. Then

$$L(P_n, f) = \sum_{i=1}^{n} m_i \Delta x_i. \quad \text{and}$$

$$U(P_n, f) = \sum_{i=1}^{n} M_i \Delta x_i.$$

write $\varphi_n = \sum_{i=1}^{n} m_i \chi_{[x_i, x_{i+1})}$,

then $\varphi_n \uparrow f$ and $\int_0^1 \varphi_n \, dx = L(P_n, f)$.

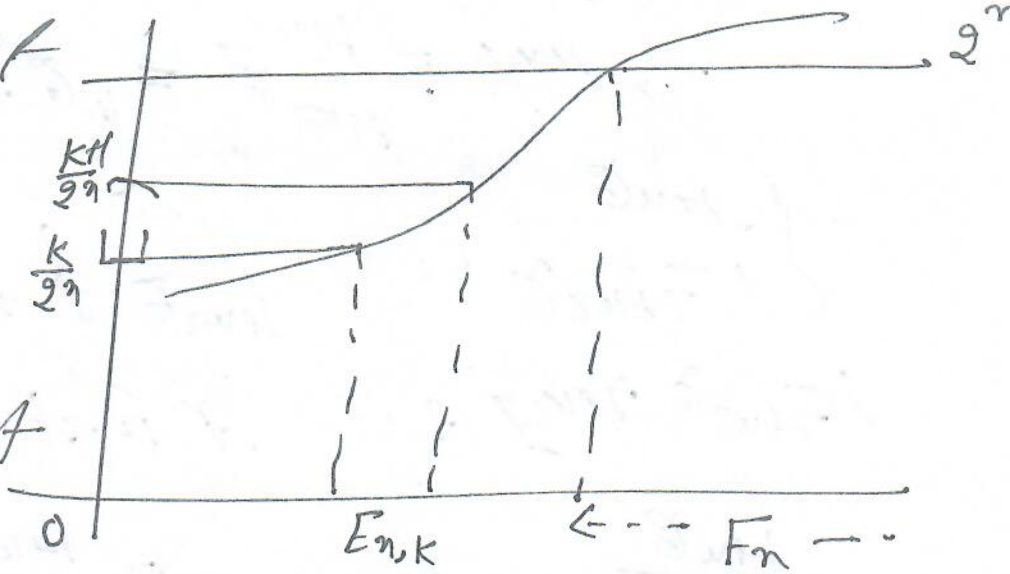
and $\lim \int_0^1 \varphi_n \, dx = \lim L(P_n, f) = \int_a^b f(x) dx$.

i.e every R-integrable function is limit of simple functions!

Hence, we can think of similar conclusion for a measurable function.

Theorem: Let $f : (X, S) \longrightarrow [0, \infty]$ be measurable. Then $\exists$ a sequence φ_n of simple functions on X such that
(i) $\varphi_n \leq f$ and $\varphi_n \uparrow f$ pointwise
(ii) $\varphi_n \uparrow f$ uniformly on any set $A \subset X$ when f is bounded.

Proof: We first divide the image of f into $[0, 2^n)$ into 2^{2n} disjoint parts.



Let $F_n = \{x : f(x) \geq 2^n\}$ and
$E_{n,K} = \{x : \frac{K}{2^n} \leq f(x) < \frac{K+1}{n}\}$.

Define $$\varphi_n = \sum_{K=0}^{2^{2n}-1} \frac{K}{2^n} \chi_{E_{n,K}} + 2^n \chi_{F_n}$$

Then $\varphi_n \geq 0$ and $E_{n,K}$'s are disjoint measurable sets in X.

(i) φ_n is an increasing sequence

Claim $\varphi_n(x) \leq \varphi_{n+1}(x)$, $\forall\, x \in X$, $\forall\, n \geq 1$.

If $x \in E_{n,k} = E_{n+1,2k} \cup E_{n+1,2k+1}$.

For $x \in E_{n+1,2k}$, $\varphi_n(x) = \frac{k}{2^n} = \frac{2k}{2^{n+1}} = \varphi_{n+1}(x)$.

For $x \in E_{n+1,2k+1}$, $\varphi_n(x) = \frac{k}{2^n} < \frac{2k+1}{2^{n+1}} = \varphi_{n+1}(x)$.

Now, if $x \in F_n = (F_n \setminus F_{n+1}) \cup F_{n+1}$. Then

for $x \in F_{n+1}$, $\varphi_n(x) = 2^n < 2^{n+1} = \varphi_{n+1}(x)$.

For $x \in F_n \setminus F_{n+1}$, we have

$$2^n = \frac{2^{2n+1}}{2^{n+1}} \le f(x) < 2^{n+1} = \frac{2^{2n+2}}{2^{n+1}}.$$

$$\Rightarrow x \in E_{n+1,2^{2n+1}} \cup \cdots \cup E_{n+1,2^{2n+2}-1}$$

Then $\varphi_{n+1}(x) \in \left\{ \frac{2^{2n+1}}{2^{n+1}}, \ldots, \frac{2^{2n+2}-1}{2^{n+1}} \right\}$.

Hence,

$$\varphi_n(x) = 2^n = \frac{2^{2n+1}}{2^{n+1}} \le \varphi_{n+1}(x).$$

That is, $\varphi_n \uparrow$ & $\varphi_n \le f$.

(iii) $\varphi_n \to f$ point-wise.

If $f(x) = \infty$, for some $x \in X$, then

$$\{x : f(x) = \infty\} = \bigcap \{x : f(x) \ge 2^n\}$$

$$\Rightarrow \varphi_n(x) = 2^n \to \infty = f(x).$$

Now,

$$\{x : f(x) < \infty\} = \bigcup_{n=1}^{\infty} \{x : f(x) < 2^n\}.$$

If $f(x) < \infty$, then $\exists\, n_0 = n_0(x) \in \mathbb{N}$ s.t

$$f(x) < 2^{n_0} < 2^n, \quad \forall\, n \geq n_0.$$

$\Rightarrow x \in E_{n,k}$ for some k. Hence

$$\varphi_n(x) = \frac{k}{2^n} \text{ and } \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}$$

$$\Rightarrow \quad 0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}, \quad \forall\, n \geq n_0(x).$$

$\Rightarrow \varphi_n \to f$ pointwise

(iv) Let $A = \{x \in X : f(x) \leq M\}$. Then

$\exists\, n_0 \in \mathbb{N}$ s.t $\quad f(x) < 2^n, \ \forall\, n \geq n_0, \ \forall x \in A$

Hence $\quad 0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}, \ \forall\, n \geq n_0.$

$$\Rightarrow \sup_{x \in A} \{f(x) - \varphi_n(x)\} \leq \frac{1}{2^n}, \ \forall\, n \geq n_0.$$

$\Rightarrow \varphi_n \to f$ unif. on A.

<u>Corollary</u>: If $f : (X, S) \longrightarrow \overline{\mathbb{R}}$ is measble. Then $\exists$ a seqn φ_n of simple functions on X s.t. $\varphi_n \longrightarrow f$ p.w. & $|\varphi_n| \uparrow |f|$ pointwise.

Proof: Let $f = f^+ - f^-$, then f^+, f^- are measurable and $f^+, f^- : (X, S) \longrightarrow [0, \infty]$.

Hence, $\exists$ $\varphi_n^+ \uparrow f^+$ & $\varphi_n^- \uparrow f^-$. (95)

Let $\varphi_n = \varphi_n^+ - \varphi_n^-$. Then $\varphi_n \longrightarrow f$ p.w.

and $|\varphi_n| = \varphi_n^+ + \varphi_n^- \uparrow |f|$.

Q How far uniform conv. is from pointwise convergence?

Ex. Let $f_n(t) = t^n$, $t \in [0,1]$. Then

$f_n(t) \to 0$ if $0 \leq t < 1$ & $f_n(1) \to 1$.

$\therefore$ $\sup_{0 \leq t \leq 1} |f_n(t) - f(t)| = 1 \not\to 0$ as $n \to \infty$.

Hence, f_n is not unif conv. However, for $\forall \epsilon > 0$, $f_n \to 0$ unif on $[0, 1-\epsilon]$ and

$m([1-\epsilon, 1]) = \epsilon$.

In fact any discontinuous function can be thought as limit of seq^n of continuous f^ns. (a consequence of Lusin's theorem, we see it later).

Hence, the above excercise can be generalized as follows. This is known as Egorov's theorem.

Egorov's theorem: Let (X, S, μ) be a finite measure space. Let f_n be a seqn of mble functions on X which converges to f pointwise. Then for each $\epsilon > 0$, $\exists$ a mble set $E \subset X$ s.t $\mu(E) < \epsilon$ and the sequence f_n converges to f uniformly on E^c. (96)

Proof: Idea of the proof is to collect all those points where unif conv. fails. This construction is based on the following ~~obsesrevation~~ observation.

(i) $\forall \epsilon > 0$, $\exists n_0 \in N$ st $|a_n - a| < \epsilon$ is equivalent that $\forall k \in N$, $\exists n_0 \in N$ s.t $|a_n - a| < \epsilon$.

(ii) $f_n(x) \to f(x)$ pointwise if $\forall \frac{\epsilon}{k} > 0$, $\exists n_0 = n_0(x)$ s.t $|f_n(x) - f(x)| < \frac{\epsilon}{k}$, $\forall n \geq n_0 = n_0(x)$.

For unif conv. $\sup_{x \in X} n_0(x) < \infty$.

(iii) $f_n \to f$ unif on X if $\forall \frac{\epsilon}{k} > 0$, $\exists n_0 \in N$ s.t $|f_n(x) - f(x)| < \frac{1}{k}$, $\forall n \geq n_0$.

Hence, if f_n does not converge to f unif on X then for some $k \in N$, $\nexists n_0 \in N$ s.t
$$|f_n(x) - f(x)| < \frac{1}{k}, \forall x \in X, \forall n \geq n_0.$$

i.e for some $k \in N$, and $\forall n_0 \in N$, $\exists x \in X$ s.t $|f_n(x) - f(x)| \geq \frac{1}{k}$ for ctbly many $n \geq$

Hence without loss of generality, we collect all points in X s.t. $\forall K \in \mathbb{N}$, and $\forall n_0 \in \mathbb{N}$

$$|f_n(x) - f(x)| \geq \frac{1}{K}, \quad \forall n \geq n_0.$$ (97)

Let $E_{m,K} = \bigcup_{n=m}^{\infty} \{x \in X : |f_n(x) - f(x)| \geq \frac{1}{K}\}$. Then for each fixed K, $E_{m,K} \downarrow$ sequence in S, and $\mu(E_{m,K}) < \mu(X) < \infty$. Hence,

$$(*) \quad \lim_{m \to \infty} \mu(E_{m,K}) = \mu\left(\bigcap_{m=1}^{\infty} E_{m,K}\right) = \mu(\phi) = 0.$$

(if $x \in \bigcap E_{m,K}$, then $|f_{n_j}(x) - f(x)| \geq \frac{1}{K}, \forall j \geq 1$)

$$\Rightarrow |f(x) - f(x)| \geq \frac{1}{K}.)$$

From $(*)$, for $\epsilon > 0$, $\exists$ $M_K \in \mathbb{N}$ s.t.

$$\mu(E_{m,K}) < \frac{\epsilon}{2^K}, \quad \forall m \geq M_K.$$

Let $E = \bigcup_{K=1}^{\infty} E_{M_K,K}$. Then $\mu(E) \leq \epsilon$.

Now, for $x \in E^c = \bigcap_{K=1}^{\infty} E^c_{M_K,K} \Rightarrow x \in E^c_{M_K,K}, \forall K \geq 1$

$$\Rightarrow |f_m(x) - f(x)| < \frac{1}{K}, \ \forall K \geq 1 \ \& \ \forall m \geq M_K.$$

Hence, for each $K \in \mathbb{N}$, $\sup_{x \in E^c} |f_m(x) - f(x)| \leq \frac{1}{K}$, $\forall m \geq M_K$.

Thus, $f_m \to f$ unif on E^c.

Remark: Finiteness of (X, S, μ) is necessary for the Egorov's theorem. For this, let

$$f_n : (\mathbb{R}, M, m) \longrightarrow \mathbb{R} \text{ and}$$

$$f_n = \chi_{[n, n+1]}, \quad n \in \mathbb{N}.$$

Then for each $x \in \mathbb{R}$, $\exists$ $n_0 = n_0(x)$ s.t

$$x \in [n_0, n_0+1] \text{ and } f_n(x) = 0, \ \forall n > n_0.$$

Thus, $f_n \to 0$ point wise on $\mathbb{R}$. However, it fails to follow Egorov's theorem. For any set $E \subset [n, n+1]$, with $0 < m(E) < \epsilon$,

$$\sup_{x \in E^c \subset \mathbb{R}} |f_n(x) - f(x)| = 1 \not\to 0.$$

Ex. Show that $f_n = \frac{1}{n} \chi_{(0,n)}$ converges unif to 0.

Ex. Show that $f_n = n\chi_{[0, \frac{1}{n}]}$ converges p.w. a.e to 0. but not p.w.

(Hint: $f_n(0) = n \to \infty$, $f_n(x) = 0$, for $x \geq 1$ for $n \geq 2$. and if $0 < x < 1$, $\exists$ $n_0 \in \mathbb{N}$ $0 < x < \frac{1}{n_0}$)

Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \chi_{\mathbb{R} \setminus \mathbb{Q}}(x)$.

Then f is nowhere continuous but

$$f|_{\mathbb{R} \setminus \mathbb{Q}} = 1 \ \ \& \ \ m(\mathbb{Q}) = 0 < \epsilon.$$

i.e. f is cont on $\mathbb{R} \setminus \mathbb{Q}$ & $m(\mathbb{Q}) = 0 < \epsilon$.

We shall show that every m'ble function is nearly continuous (Lusin's thm)

__Lemma:__ Let $E = \overset{n}{\underset{i=1}{\dot{\cup}}} E_i$, $E_i \in M(\mathbb{R})$ and define $\varphi : E \to \mathbb{R}$ by $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$. Then for each $\epsilon > 0$, $\exists$ a closed set $F \subset E$ s.t. $\varphi/_F$ is cont & $m(E \setminus F) < \epsilon$.

__Proof:__ Since $E_i \in M$, $\forall \epsilon > 0$, $\exists$ a closed set $F_i \in \mathbb{R}$ s.t. $F_i \subseteq E_i$ and

$$m(E_i \setminus F_i) < \frac{\epsilon}{n}, \quad \forall i = 1, 2, \ldots, n.$$

Let $F = \overset{n}{\underset{i=1}{\dot{\cup}}} F_i$. Then

$$m(E \setminus F) = m(\dot{\cup}_i (E_i \setminus F))$$

$$= \sum m(E_i \setminus F) \leq \sum m(E_i \setminus F_i) < \epsilon.$$

Then $\varphi/_F$ is cont. For $x_m, x \in F = \overset{n}{\underset{i=1}{\dot{\cup}}} F_i$, and $x_m \to x$, $\exists\, m_0 \in \mathbb{N}$ s.t.

$$x_m \in F_{i_0}, \quad \forall\, m \geq m_0, \text{ for some } i_0.$$

Hence $x \in F_{i_0}$ ($\because F_{i_0}$ is closed)

$\therefore \varphi(x_m) = \alpha_{i_0} = \varphi(x) \Rightarrow \lim \varphi(x_m) = \varphi(x)$,

Hence $\varphi/_F$ is continuous.

Cor: If $m(E) < \infty$, then $\forall \epsilon > 0$, $\exists$ a compact set $K \subset E$ s.t $f|_K$ is continuous and $m(E \setminus K) < \epsilon$.

Lusin's theorem: Let $E \subset M(\mathbb{R})$ and f is Lebesgue m'ble on E. Then for each $\epsilon > 0$, $\exists$ a closed set $F \subset E$ s.t $f|_F$ is cont and $m(E \setminus F) < \epsilon$.

Proof: We prove the result in two steps.

(i) Let $m(E) < \infty$. Let φ_n be a seqn of simple functions that converges to f p.w.

Then by the previous lemma, for each $\epsilon > 0$, $\exists$ a m'ble set $E_n \subset E$ s.t $m(E_n) < \frac{1}{3}\frac{\epsilon}{2^n}$ and $\varphi_n|_{E \setminus E_n}$ is continuous.

By Egorov's theorem, for each $\epsilon > 0$, $\exists$ a m'ble set F s.t $m(E \setminus F) < \epsilon/3$ and

$$\varphi_n \longrightarrow f \text{ unif. on } F.$$

Let $G = F \setminus \bigcup_{n=1}^{\infty} E_n$, then $\varphi_n|_G$ is cont and $\varphi_n \to f$ unif on G. Then $f|_G$ is continuous. (being uniform limit of seqn of continuous f'ns is continuous).

Since $G \in \mathcal{M}(\mathbb{R})$ & $m(G) \leq m(E) < \infty$, for each $\epsilon > 0$, $\exists$ a compact set $K \subset G$ s.t. $m(G \setminus K) < \epsilon/3$. (101)

Now, $m(E \setminus K) = m(E \setminus G) + m(G \setminus K)$.

But $m(E \setminus G) = m(E \setminus F) + m(F \setminus G)$

$$< \epsilon/3 + m(\cup E_n) < \frac{2\epsilon}{3}.$$

Hence, $m(E \setminus K) < \epsilon$ & $f/_K$ is cont.

(ii) Let $m(E) = \infty$. Then $E = \bigcup_{n \in \mathbb{Z}} (E \cap [n, n+1))$.

Let $E_n = E \cap [n, n+1)$. Then for $m(E_n) < \infty$, by finite case, $\forall \epsilon > 0$, $\exists$ cpt $K_n \subset E_n$ s.t. $f/_{K_n}$ is cont & $m(E_n \setminus K_n) < \frac{\epsilon}{2^{|n|+1}}$.

Let $F = \dot\cup K_n$, $K_n \subset [n, n+1)$. Then F is closed set! (as we have seen earlier).

Define $g : F \to \mathbb{R}$ by $g = \sum_{n \in \mathbb{Z}} f \chi_{K_n}$.

Then g is continuous on F.

Let $x, x_k \in F$ & $x_k \to x$. Then $\exists$ $k_0 \in \mathbb{N}$ s.t. $x_k \in K_{n-1} \dot\cup K_n$, $\forall k \geq k_0$.

($\because$ for $\epsilon = 1/2$, $\exists$ $k_0 \in \mathbb{N}$ s.t.

$x_k \in (x-\frac{1}{2}, x+\frac{1}{2}) \subset [n-1, n) \dot\cup [n, n+1)$ for some $n \in \mathbb{N}$)

Since $x_k \to x$, $x \in K_{n-1}$ & K_n. Thus,

$g(x_k) = f(x_k) \to f(x) = g(x)$. (102)

Now, $m(E \setminus F) \leq \epsilon$.

Question: Does the converse of Egorov's thm true?

Littlewood's three principles:

(i) Every set is nearly finite union intervals.

(ii) Every function is nearly continuous.

(iii) Every convergent sequence is nearly uniformly convergent.

Here (i) means that if $E \in \mathcal{M}(\mathbb{R})$ & $m(E) < \infty$. Then for each $\epsilon > 0$, $\exists$ $O = \dot\bigcup_{n=1}^{N} I_n$ s.t.

$$m(O \Delta E) < \epsilon.$$

For $\epsilon > 0$, $\exists$ $O \supset E$ st $m(O \setminus E) < \epsilon$. But then $m(O) = \sum m(I_n) < \infty$, $O = \dot\bigcup I_n$

For $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ st $\sum_{n=N+1}^{\infty} m(I_n) < \epsilon/2$.

Let $O' = \dot\bigcup_{n=1}^{N} I_n$, and $O'' = \dot\bigcup_{n=N+1}^{\infty} I_n$. Then

$m(O' \Delta E) = m(O' \setminus E) + m(E \setminus O') < \epsilon/2 + m(O'' \setminus O') < \epsilon$