Lebesgue Integration :

Let (X, S, μ) be a measure space.

Let $\varphi : X \longrightarrow [0,\infty]$ s.t

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}, \quad E_i \in S, \quad \alpha_i \in [0,\infty].$$

We write $\int_X \varphi \, d\mu := \sum_{i=1}^{n} \alpha_i \mu(E_i)$ (adoptation)

Notice that $\int_X \varphi \, d\mu = 0$ iff $\varphi = 0$ a.e.

$$\left(\because \sum_{i=1}^{n} \alpha_i \mu(E_i) = 0 \iff \alpha_i \mu(E_i) = 0, \ \forall \, i = 1,2,\ldots n. \right.$$

$$\left. \iff \alpha_i = 0 \text{ or } \mu(E_i) = 0 \right).$$

Now, if $E \in S$, then

$$\varphi|_E = \sum_{i=1}^{n} \alpha_i \chi_{E_i \cap E}. \quad \text{if } \{E_i\}_{i=1}^{n}$$

is a family of pairwise disjoint sets.

Then $\int_E \varphi \, d\mu = \int_X \varphi \chi_E \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i \cap E).$

Ex. Let $f : (\mathbb{R}, M, m) \to \mathbb{R}$ be defined by

$f = \chi_{\mathbb{Q}}$. Then $f = 0 \cdot \chi_{\mathbb{R} \setminus \mathbb{Q}} + 1 \cdot \chi_{\mathbb{Q}}$

$\Rightarrow \int_{\mathbb{R}} f \, dm = 0 \cdot \infty + 1 \cdot 0 = 0$ if

we adopt $0 \cdot \infty = 0 = \infty \cdot 0$.

Let φ be a simple function on a measure space (X, S, μ) to $[0, \infty]$. Then

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}, \quad E_i \cap E_j = \phi \text{ if } i \neq j$$

and $E_i \in S$. By assigning zero out side $\bigcup_{i=1}^{n} E_i$, we may assume that

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i} \ \& \ \bigcup_{i=1}^{n} E_i = X.$$

Let $L^+(X, S, \mu)$ be space of all S-m'ble f's $f: X \to [0, \infty]$.

<u>Proposition</u>: Let φ, ψ be two simple functions in $L^+(X, S, \mu)$. Then,

(i) For $c \geq 0$, $\int_X c\varphi \, d\mu = c \int_X \varphi \, d\mu$.

(ii) $\int_X (\varphi + \psi) \, d\mu = \int_X \varphi \, d\mu + \int_X \psi \, d\mu$, linearity.

(iii) if $\varphi \leq \psi$, then $\int_X \varphi \, d\mu \leq \int_X \psi \, d\mu$,

(iv) if $\nu: S \to [0, \infty]$ be defined by

$$\nu(A) = \int_A \varphi \, d\mu, \quad \text{for } A \in S, \text{ then}$$

ν is a measure on (X, S).

<u>Proof</u> (i) Proof of it is trivial.

(ii) Let $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ and $\psi = \sum_{j=1}^{m} \beta_j \chi_{F_j}$.

w.l.g. we can write $X = \dot{\bigcup}_{i=1}^{n} E_i$ & $X = \dot{\bigcup}_{j=1}^{m} F_j$.

Then $E_i = \dot{\bigcup}_{j=1}^{m} (E_i \cap F_j)$ and $F_j = \dot{\bigcup}_{i=1}^{n} (E_i \cap F_j)$.

Now,

$$\int_X \varphi + \int_X \psi = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \mu(E_i \cap F_j) + \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j \mu(E_i \cap F_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) \mu(E_i \cap F_j) \qquad \text{—— (1)}$$

and $\int_X (\varphi + \psi) d\mu = \int_X \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) \chi_{(E_i \cap F_j)} d\mu$.

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) \mu(E_i \cap F_j)$$

$$= \int_X \varphi \, d\mu + \int_X \psi \, d\mu \quad (\text{by (1)}).$$

(iii) Since $\varphi \le \psi$, we get $\alpha_i \le \beta_j$ if $E_i \cap F_j \ne \phi$.

(for this, let $\varphi(x) \le \psi(x) \Rightarrow x \in E_i$ & $x \in F_j$ for some i, j).

$$\int_X \varphi \, d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \mu(E_i \cap F_j) \le \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j \mu(E_i \cap F_j) = \int_X \psi \, d\mu$$

(iv) For $A \in S$, write $\nu(A) = \int_A \varphi \, d\mu$. Then

$\nu(\phi) = 0$.

If $A, B \in S$ and $A \subset B$, then $\chi_A \le \chi_B$.

Hence $\varphi|_A \le \varphi|_B \Rightarrow \int_A \varphi \le \int_B \varphi \Rightarrow \nu(A) \le \nu(B)$.

Let $\{A_k\}_{k=1}^{\infty} \subset S$ & $A_k \cap A_l = \phi$, $\forall\, k \ne l$, and write $A = \bigcup_{k=1}^{\infty} A_k$. Then. (106)

$$\nu(A) = \int_A \varphi\, d\mu = \sum_{i=1}^{m} \alpha_i\, \mu(A \cap E_i)$$
$$= \sum_{k=1}^{\infty} \sum_{i=1}^{m} \alpha_i\, \mu(A_k \cap E_i)$$
$$= \sum_{k=1}^{\infty} \nu(A_k).$$

Now, it is obvious that if $\alpha, \beta \in \mathbb{R}$, then

$$\int_X (\alpha\varphi + \beta\psi)\, d\mu = \alpha \int_X \varphi\, d\mu + \beta \int_X \psi\, d\mu,$$ for φ & ψ are simple measble functions on (X, S, μ) to $[0, \infty]$.

Notice that if $\varphi \le \psi$ a.e, then $\int_X \varphi \le \int_X \psi$.

Let $E = \{x \in X : \varphi(x) > \psi(x)\}$, then $\mu(E) = 0$.

$$\int_X \varphi = \int_E \varphi + \int_{E^c} \varphi = 0 + \int_{E^c} \psi = \int_E \psi + \int_{E^c} \psi = \int_X \psi.$$

$(\because \int_E \varphi = 0$ iff $\varphi = 0$ a.e$)$

Next, consider $f \in L^+(X, S, \mu)$, then $\exists$ a seq^n of simple f's $\varphi_n \uparrow f$ p.w. Hence $\int_X \varphi_n\, d\mu \uparrow$ sequence in $[0, \infty]$. Thus,

$$\lim_{n\to\infty} \int_X \varphi_n\, d\mu = \sup_{n \ge 1} \int_X \varphi_n\, d\mu.$$ (Important)

We define, for $f \in L^+(X, S, \mu)$,

$$\int_X f\, d\mu = \sup\left\{ \int_X \varphi\, d\mu : 0 \le \varphi \le f, \ \varphi \text{ is simple and m'ble.} \right\}$$

If $f, g \in L^+(X, S, \mu)$ & $f \le g$, then

$$\int_X f\, d\mu = \sup_{0 \le \varphi \le f} \int_X \varphi\, d\mu \le \sup_{0 \le \varphi \le g} \int_X \varphi\, d\mu = \int_X g\, d\mu.$$

Further, if $f : (X, S, \mu) \xrightarrow{\text{m'ble}} \bar{\mathbb{R}} = [-\infty, \infty]$, then $f = f^+ - f^-$. We write

$$\int_X f\, d\mu = \int_X f^+\, d\mu - \int_X f^-\, d\mu.$$ if at least one of $\int_X f^+ d\mu$ or $\int_X f^- d\mu$ is finite.

<u>Lemma</u>: Let $f \in L^+(X, S, \mu)$. Then

$$\int_X f\, d\mu = 0 \text{ iff } f = 0 \text{ a.e. } \mu.$$

Proof: Suppose $f = 0$ a.e. If $0 \le \varphi \le f$, φ is simple m'ble fn, then $\varphi = 0$ a.e.

then $\int_X f\, d\mu = \sup_{\varphi \le f} \int_X \varphi\, d\mu = 0$ (by previous resu

Next, let $E = \{x \in X : f(x) > 0\}$. Then

$$E = \bigcup_{n=1}^{\infty} \{x \in X : f(x) \ge \tfrac{1}{n}\} = \bigcup_{n=1}^{\infty} E_n.$$

Now, $m(E_n) = n \int_{E_n} \frac{1}{n}\, d\mu \le n \int_{E_n} f\, d\mu \le n \int_X f\, d\mu = 0$

Monotone Convergence theorem (MCT):

Let $f, f_n \in L^+(X, S, \mu)$ be such that $f_n \uparrow f$ p.w, then $\int_X f\,d\mu = \lim \int_X f_n\,d\mu$.

Proof: Since $f_n \leq f_{n+1} \leq f$, the limit of $\int_X f_n$ will be bounded above by $\int_X f$. Hence, $\lim \int_X f_n \leq \int_X f$.

In order to show other inequality it is enough to show that for each $\epsilon > 0$,

$$\lim \int_X f_n \geq (1-\epsilon) \int_X f \quad \text{or for } \varphi \leq f,$$

$$\lim \int_X f_n \geq (1-\epsilon) \int_X \varphi.$$

Let $E_n = \{ x \in X : f_n(x) \geq (1-\epsilon)\varphi(x) \}$. Since $f_{n+1} \geq f_n \Rightarrow E_n \subset E_{n+1}$. Moreover, $X = \bigcup_{n=1}^{\infty} E_n$. For this, let $x \in X \setminus \bigcup_{n=1}^{\infty} E_n$, then

$$f_n(x) < (1-\epsilon)\varphi(x), \quad \forall\, n \geq 1$$

$$\Rightarrow f(x) \leq (1-\epsilon)\varphi(x), \text{ is a contradiction}$$

Let $\nu(E_n) = \int_{E_n} \varphi$. Then ν becomes a measure on (X, S) and $E_n \uparrow X$. Hence,

$$\lim \nu(E_n) = \nu(X). \quad \text{Thus,}$$

$$(1-\epsilon)\int_X \varphi = \lim \int_{E_n} (1-\epsilon)\varphi \leq \lim \int_{E_n} f_n \leq \lim \int_X f_n.$$

Remark: f_n increases to f p.w. is necessary in MCT. (109)

Let $f_n : (\mathbb{R}, \mathcal{M}, m) \to \mathbb{R}$ by

$$f_n = \frac{1}{n}\chi_{[0,n]}. \text{ Then } f_n \to 0 \text{ p.w.}$$

(even $f_n \to 0$ unif too). But

$$\lim \int_{\mathbb{R}} f_n \, dm = 1 \neq 0 = \int_{\mathbb{R}} \lim f_n \, dm.$$

Exercise: Verify MCT for $f_n : \mathbb{R} \to [0,\infty]$, given by (i) $f_n = \chi_{(n, n+1)}$

(ii) $f_n = n\chi_{(0, \frac{1}{n})}$.

Corollary to MCT: Let $f, f_n \in L^+(X, S, \mu)$ be such that $f_n \uparrow f$ p.w. a.e on X. Then

$$\int_X f \, d\mu = \lim \int_X f_n \, d\mu.$$

Proof: Let $f_n \to f$ p.w. on $E \subset X$. Then $\mu(E^c) = 0$. Hence $E, E^c \in S$. By MCT on measure space $(E, S \cap E, \mu)$, we get

$$\int_E f \, d\mu = \lim \int_E f_n \, d\mu \Rightarrow \int_X \chi_E f \, d\mu = \lim \int_X \chi_E f_n \, d\mu$$

Now, $$\int_X f = \int_X (\chi_E f + \chi_{E^c} f) = \int_X \chi_E f + \int_X \chi_{E^c} f$$

(by linearity of integration)

$$= \lim \int_X \chi_E f_n + \lim \int_X \chi_{E^c} f_n .$$

Remark: 1. Integration is linear on $L^+(X, S, \mu)$. That is, $f \mapsto \int_X f \, d\mu$ is a linear map.

Let $f, g \in L^+(X, S, \mu)$. Then $\exists$ seq^{n}s φ_n & ψ_n of simple msble f^ns in $L^+(X, S, \mu)$, such that $\varphi_n \uparrow f$ p.w. & $\psi_n \uparrow g$ p.w.

By MCT,

$$\int_X (f+g) \, d\mu = \lim \int_X (\varphi_n + \psi_n) \, d\mu = \lim \int_X \varphi_n \, d\mu + \lim \int_X \psi_n \, d\mu$$

$$= \int_X f + \int_X g .$$

Remark 2. Let φ_n be a seq^n of simple f^n in $L^+(X, S, \mu)$ s.t. $\varphi_n \to f \in L^+(X, S, \mu)$. Then $\psi_n = \varphi_n \chi_{[-n, n]} \longrightarrow f$, where $\int_X \psi_n < \infty$.

Consider $f \in L^+(X, S, \mu)$ and $E \in S$. Then the set function $E \longmapsto \int_E f \, d\mu$ defines a measure on (X, S). This will be followed by the following equivalent statement of MCT, known as Beppo-Levi theorem.

<u>Theorem</u>: Let $f_n \in L^+(X, S, \mu)$. Then

$$\int_X \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_X f_n .$$

Proof: Notice that $\sum_{k=1}^{n} f_k \uparrow \sum_{k=1}^{\infty} f_k$. Hence

$$\int_X \sum_{k=1}^{\infty} f_k = \int_X \lim_{n\to\infty} \sum_{k=1}^{n} f_k = \lim_{n\to\infty} \int_X \sum_{k=1}^{n} f_k = \lim_{n\to\infty} \sum_{k=1}^{n} \int_X f_k$$

(by MCT)

Now, let $\nu(E) = \int_E f \, d\mu$. Then $\nu(\phi) = 0$.

If $E = \dot{\bigcup}_{n=1}^{\infty} E_n$, $E_n \in S$, then $f|_E = \sum_{n=1}^{\infty} \chi_{E_n} f$!

By Beppo-Levi thm,

$$\nu(E) = \sum_{n=1}^{\infty} \int_X \chi_{E_n} f = \sum_{n=1}^{\infty} \nu(E_n).$$

Hence, ν is a measure on (X, S).

Recall that monotone conv. thm (MCT) allow us to commute limit & integral, while seq^n f_n is $\uparrow$ & non-negative etc. However, other cases still need to address. For example, if $f_n = \frac{1}{n} \chi_{(n, n+1)}$ on $(\mathbb{R}, M, m)$, then $f_n \to 0$ but f_n is not monotone $\uparrow$. and

$$\lim \int f_n \, dm = 0 = \int \lim f_n.$$

If $f_n = \frac{1}{n} \chi_{(0,n)} \to 0$ but $\int f_n \to 1 > 0 = \int \lim f_n$.

Hence, "equality" need not be the case for arbitrary seq^n, but we can compare both the limit's, i.e, $\lim \int f_n$ and $\int_X \lim f_n$.

Fatou's Lemma: Let $f_n \in L^+(X, S, \mu)$. Then (112)

$$(*) : \int_X \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_X f_n .$$

Proof: Since $\inf_{n\geq k} f_n \leq f_j$, $\forall j \geq k$,

$$\Rightarrow \int_X \inf_{n\geq k} f_n \leq \int_X f_j , \quad \forall j \geq k.$$

$$\Rightarrow \int_X \inf_{n\geq k} f_n \leq \inf_{j\geq k} \int_X f_j \quad \text{—(1)}$$

Now, let $g_k = \inf_{n\geq k} f_n = \inf\{f_k, f_{k+1}, \dots\}$. Then

$$g_k \uparrow \sup_{k\geq 1} \left(\inf_{n\geq k} f_n\right) = \liminf_{k\to\infty} f_k .$$

Hence, by MCT, it follows that

$$\int_X \liminf_{k\to\infty} f_k = \int_X \lim_{k\to\infty} g_k = \lim_{k\to\infty} \int_X g_k$$

$$\leq \lim_{k\to\infty} \inf_{j\geq k} \int_X f_j$$

$$= \lim_{k\to\infty} \inf \int_X f_k .$$

Remarks (i) If $\lim_{k\to\infty} f_k$ and $\lim_{k\to\infty} \int_X f_k$ both exist, then $\int_X \lim f_k \leq \lim \int_X f_k$.

(ii) Inequality in $(*)$ can be strict. For this, let $f_n = \frac{1}{n} \chi_{[0,n]}$ on $(\mathbb{R}, \mathcal{M}, m)$. Then, we get

$$\int_{\mathbb{R}} \liminf f_n = \int_{\mathbb{R}} \lim f_n = 0 < 1 = \liminf \int_X f_n.$$ (113)

(iii) Fatou's lemma need not be true beyond non-negative functions. For example,

$$f_n = -\frac{1}{n}\chi_{[n,2n]} \text{ on } (\mathbb{R}, M, m).$$

Here $\lim\limits_{k\to\infty} \inf\limits_{n\geq k} f_n = \lim\limits_{k\to\infty}\left(-\frac{1}{k}\right) = 0$, however,

$$\int_X \lim_{k\to\infty}\inf_{n\geq k} f_n = 0 > -1 = \lim_{k\to\infty}\inf_{n\geq k}\int_X f_n.$$

Ex. Let $f_n \in L^+(X, S, \mu)$ and $f = \lim f_n$ with $f_n \leq f$, $\forall n \geq 1$. Show that

$$\int_X \lim f_n = \lim \int_X f_n.$$

(Hint: Use Fatou's lemma for f_n & $f - f_n \geq 0$ both)

Ex. Let $f_n \in L^+(X, S, \mu)$ be given by

$$f_n(x) = \begin{cases} \chi_E(x) & \text{if } n \text{ odd} \\ 1-\chi_E(x) & \text{if } n \text{ even}, \end{cases}$$

for some $E \in S$. Verify Fatou's lemma for f_n.

(Hint: use $\int_X f = \int_E f + \int_{E^c} f$ etc).

Integrable functions:

Let $f : (X, S, \mu) \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ be measurable. Then f^+, f^- both are measurable and $f = f^+ - f^-$, $|f| = f^+ + f^-$.

Def^n: f is said to be integrable on (X, S, μ) if both $\int_X f^+$ & $\int_X f^-$ are finite. In this case we write

$$\int_X f = \int_X f^+ - \int_X f^-.$$

As $|f| = f^+ + f^-$, it follows that $\int_X |f|$ is finite iff $\int_X f$ is finite.

Denote $L^1(X, S, \mu) := \left\{ f : X \xrightarrow{\text{measurable}} \overline{\mathbb{R}},\ \int_X |f| d\mu < \infty \right\}$.

Notice that we also use the symbol $L^1(X)$ or $L^1(X, S)$. We can see that

$R[0,1] \subsetneq L^1([0,1], M, m)$. Since for

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ -1 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1], \end{cases} \quad |f| = 1.$$

Then $f \in L^1([0,1], M, m)$ but $f \notin R[0,1]$.

Further, if $E \in S$, & $f \in L^1(X, S, \mu)$. Then

$$\int_E f = \int f \chi_E = \int f_E^+ - \int_E f^-.$$

Notice that $L^1(X,S,\mu)$ is a linear space. We define a norm on $L^1(X,S,\mu)$. For this, we need to recognize $f=0$ a.e to be $f=0$. Since

$$\int_X |f| = 0 \text{ iff } |f|=0 \text{ a.e. iff } f=0 \text{ a.e.}$$

Hence, write $\|f\|_1 = \int_X |f|$, if $f \in L^1(X)$.

<u>Lemma</u>: Let $f \in L^1(X)$. Then

(i) $\left|\int_X f\right| \le \int_X |f|$, (Conti. of $f \mapsto \int_X f$)

(ii) $\int_E f = 0$ for all $E \in S$ iff $\int_X |f| = 0$,

iff $f=0$ a.e,

(iii) $\{x \in X : f(x) \neq 0\}$ is a σ-finite set.

(iv) $\mu\{x \in X : |f(x)| = \infty\} = 0$.

<u>Proof</u>: (i) $\left|\int_X f\right| = \left|\int_X f^+ - \int_X f^-\right| \le \int_X f^+ + \int_X f^- = \int_X |f|$.

(ii) Suppose $\int_E f = 0$, $\forall E \in S$. Let

$E_0 = \{x \in X : f(x) \geq 0\}$. Then $E_0 \in S$, and

$$\int_X |f| = \int_{E_0} f + \int_{E_0^c} -f = \int_{E_0} f - \int_{E_0^c} f = 0.$$

If $\int_X |f| = 0$, then for $E \in S$,

$$\left|\int_E f\right| = \left|\int_X \chi_E f\right| \leq \int_X |\chi_E f| \leq \int_X |f| = 0.$$

(iii) $\{x \in X : f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \{x \in X : |f(x)| \geq \frac{1}{n}\} = \bigcup_{n=1}^{\infty} E_n$

Now, $\mu\{x : |f(x)| \geq \frac{1}{n}\} = n \int_X \frac{1}{n} \chi_{E_n} \leq n \int_X |f(x)| \chi_{E_n} < \infty$,

since $\int_X |f| < \infty$.

(iv) $\{x \in X : |f(x)| = \infty\} = \bigcap_{n=1}^{\infty} \{x : |f(x)| \geq n\} = \cap F_n$.

But $\mu(F_n) \leq \frac{1}{n} \int_X |f|$, $\forall\ n \geq 1$. Implies

$\mu\{x \in X : |f(x)| = \infty\} \leq \mu(F_n) \leq \frac{1}{n} \|f\|_1 \to 0$.

Note that in proving (iii) & (iv), we have proved the following interesting result.

<u>Chebyshev's inequality</u>:

Let $f \in L^1(X, S, \mu)$ and $\alpha > 0$. Then

$\mu\{x \in X : |f(x)| \geq \alpha\} \leq \frac{1}{\alpha} \|f\|_1$.

So far we have shown that $L^1(X, S, \mu)$ is a normed linear space. For $f \in L^1(X, S, \mu)$,

$\|f\|_1 = \int_X |f|\, d\mu < \infty$, and $\|f\|_1 = 0$

iff $f = 0$ a.e.

Next, we shall show that $L^1(X, S, \mu)$ is a complete space. For this we need a wonderful result, known as DCT. (117)

Dominated Convergence Theorem (DCT):

Let f_n be a seq^n of m'ble functions on (X, S, μ) s.t.

(i) $f_n \to f$ point wise on X

(ii) $|f_n| \leq g$, & $g \in L^1(X, S, \mu)$.

Then $\int_X f \, d\mu = \lim \int_X f_n$.

Proof: Since $\lim f_n = f$ & $|f_n| \leq g \in L^1$, it follows that $f_n, f \in L^1$, (by monotone).

Now,

$$\left.\begin{aligned} 0 \leq g + f_n &\xrightarrow{p.w.} g + f \\ 0 \leq g - f_n &\xrightarrow{p.w.} g - f \end{aligned}\right\}$$

Then by Fatou's Lemma,

$$\int_X (g+f) = \int_X \lim (g + f_n) \leq \underline{\lim} \int_X (g + f_n)$$

$$\Rightarrow \int_X f \leq \underline{\lim} \int_X f_n \quad (\because \int_X g < \infty).$$

Similarly,

$$\int_X (g-f) \leq \underline{\lim} \int_X (g - f_n)$$

$\Rightarrow \overline{\lim} \int_X f_n \leq \int_X f$. Hence, we have $\lim \int_X f_n = \underline{\lim} \int_X f_n = \int_X f$. ($\because \underline{\lim}\, a_n \leq \overline{\lim}\, a_n$)

Cor to DCT: Let f_n be a seq^n of mble function on (X, S, μ) s.t.

(i) $f_n \longrightarrow f$ point-wise a.e

(ii) $|f_n| \leq g$ a.e. for some $g \in L^1(X, S, \mu)$.

Then $\int_X f = \lim \int_X f_n$.

In the dominated conv. thm, we require, $f_n \xrightarrow{a.e} f$ p.w. & $|f_n| \leq g$ a.e, where $g \in L^1$. This implies,

$$|f_n - f| \to 0 \text{ a.e } \& \ |f_n - f| \leq 2g \in L^1.$$

Hence

$$\left|\int_X f_n - \int_X f\right| \leq \int_X |f_n - f| \longrightarrow 0, \text{ as } |f_n - f| \to 0.$$

This shows that the map $f \longmapsto \int_X f$ is "continuous" on $L^1(X, S, \mu)$.

Thus, we can think of fundamental theorem of Calculus for Lebesgue integrable function.

Theorem: Let $f \in L^1(\mathbb{R}, M, m)$. Define

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$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{[-\infty, x]} f\,dm.$$

Then F is continuous on $\mathbb{R}$ to $\mathbb{R}$.

Proof: Let $x_n, x \in \mathbb{R}$ & $x_n \to x$, then

$$f \chi_{(-\infty, x_n]} \longrightarrow f \chi_{(-\infty, x]} \quad (!).$$

By DCT, we have

$$\int f \chi_{(-\infty, x]} = \lim \int f \chi_{(-\infty, x_n]}$$

$$\Rightarrow F(x) = \lim F(x_n).$$

Ex. For $f_n : (\mathbb{R}, M, m) \xrightarrow{\text{mble}} \bar{\mathbb{R}}$ and given by:

(i) $f_n = n \chi_{[0, \frac{1}{n}]}$

& (ii) $f_n = \frac{1}{n} \chi_{[n, n+1]}$

& (iii) $f_n = \chi_{[n, n+1]}$,

$f_n \to 0$ p.w, and $\int_{\mathbb{R}} f_n\,dm = 1 > 0 = \int_{\mathbb{R}} \lim f_n\,dm.$

Hence, $|f_n| < g \in L^1$ in the statement of DCT is necessary.

Now, to prove $L^1(X, S, \mu)$ is complete, we all need to show that every absolutely conv. series in $L^1(X, S, \mu)$ is convergent.

Theorem: Let $\{f_n\}$ be a seq^n of m'ble functions on (X, S, μ) such that $\sum \int_X |f_n| < \infty$. Then $\sum_{n=1}^{\infty} f_n$ converges point wise a.e to some $f \in L^1(X, S, \mu)$, and $\int_X \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_X f_n$.

Proof: By Beppo-Levi thm,

$$\int_X \sum_{n=1}^{\infty} |f_n| = \sum_{n=1}^{\infty} \int_X |f_n| < \infty.$$

Let $g = \sum_{n=1}^{\infty} |f_n|$. Then $g \in L^1(X, S, \mu)$.

Hence, g is finite a.e on X.

$\Rightarrow \sum_{n=1}^{K} f_n \xrightarrow[a.e]{p.w.} f$ (say) ($\because$ $\forall$ abs. conv series in $\mathbb{R}$ is conv)

Since, $\left| \sum_{n=1}^{K} f_n \right| \leq \sum_{n=1}^{K} |f_n| \leq g \in L^1$, $\forall K \geq 1$.

Hence, $|f| < g \in L^1 \Rightarrow f \in L^1$.

By applying DCT to $g_K = \sum_{n=1}^{K} f_n$, we get

$$\int_X \sum_{n=1}^{\infty} f_n = \int_X \lim \sum_{n=1}^{K} f_n = \lim_{K \to \infty} \int_X \sum_{n=1}^{K} f_n = \lim_{K \to \infty} \sum_{n=1}^{K} \int_X f_n.$$

Thus, we conclude that every absolutely convergent series in $L^1(X, S, \mu)$ is convergent. Hence, $L^1(X, S, \mu)$ is a complete n.l.s.

Corollary: If $f \in L^1(X, S, \mu)$, then the set function $E \longmapsto \int_E f\,d\mu$ is countably additive. (1.21)

Proof: Let $\nu(E) = \int_E f\,d\mu$. Then $\nu(E) \in \mathbb{R}$, because, $|\nu(E)| = \left|\int_E f\,d\mu\right| \le \int_E |f|\,d\mu \le \int_X |f| < \infty$.

Next, let $E = \bigcup_{n=1}^{\infty} E_n$, & $E_n \in S$. Then

(*) $\nu(E) = \int_X f\chi_E = \int_X \sum_{n=1}^{\infty} f\chi_{E_n}$. Since,

$$\sum_{n=1}^{\infty} \int_X |f\chi_{E_n}|\,d\mu = \int_E |f|\,d\mu \le \int_X |f| < \infty.$$

Hence, by previous Thm, $\int_X \sum_{n=1}^{\infty} f\chi_{E_n}$ converges to a finite number. Thus, $\nu(E) \in \mathbb{R}$ and once again by the previous Thm,

$$\nu(E) = \sum_{n=1}^{\infty} \int_X f\chi_{E_n} = \sum_{n=1}^{\infty} \nu(E_n).$$

Note that $\nu : S \longrightarrow \mathbb{R} = (-\infty, \infty)$, which satisfies $\nu(\phi) = 0$, $\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$. Such set functions are called sign-measures which we see latter.