Theorem: Let $f:[a,\infty)\to\mathbb{R}$ be such that $f\in R[a,b]$, $\forall\, b>a$. Then $f\in L^1[a,\infty)$ iff $|f|$ is improper R-integrable. 136

Proof: Let $f\in L^1[a,\infty)$. Then $f_n=\chi_{[a,n]}f$ converges p.w. to f, and $|f_n|<|f|\in L^1$.

By DCT,
$$\infty>\int_{[a,\infty)}|f|\,dm=\lim\int|f_n|=\lim\int_a^n|f(x)|\,dx=\int_a^\infty|f(x)|\,dx.$$

i.e. $f\in R[a,\infty)$.

Conversely, suppose $|f|\in IR[a,\infty)$. Then for $g_n=\chi_{[a,n]}|f|$, $g_n\uparrow|f|$. By MCT,
$$\int|f|\,dm=\lim\int_{[a,n]}|g_n|\,dm=\lim\int_a^n|f(x)|\,dx=\int_a^\infty|f(x)|\,dx.$$

Hence $f\in L^1[a,\infty)$.

L^p-spaces:

Let (X,S,μ) be a measure space. For $1\le p<\infty$, we write
$$L^p(X,S,\mu)=\left\{f:X\xrightarrow{\text{m'ble}}\overline{\mathbb{R}} \text{ s.t. } \int_X|f|^p\,d\mu<\infty\right\}.$$

Then L^p is a linear space by identifying
$$[0]=\{g\in L^p : g=0 \text{ a.e. on } X\}.$$

Let $f,g\in L^p(X,S,\mu)$. Then

$$|f+g|^p \le (|f|+|g|)^p \le \{2\max\{|f|,|g|\}\}^p$$
$$\le 2^p \begin{cases} |f|^p & \text{if } |f| \ge |g| \\ |g|^p & \text{if } |f| \le |g| \end{cases}$$
$$\le 2^p(|f|^p + |g|^p).$$

Hence $\int |f+g|^p \le 2^p \int |f|^p + 2^p \int |g|^p < \infty$.

i.e. $f+g \in L^p$.

In general, $L^1 \not\subseteq L^2$ and $L^2 \not\subseteq L^1$.

For this, let $f(x) = \frac{1}{\sqrt{x}} \chi_{(0,1]}(x)$. Then $f \in L^1(\mathbb{R})$ but $f \notin L^2(\mathbb{R})$. Again, $g(x) = \frac{1}{1+|x|}$, $x \in \mathbb{R}$, $g \in L^2(\mathbb{R})$ but $g \notin L^1(\mathbb{R})$.

$$\int_{\mathbb{R}} |f|\, dm = 2\int_{[0,\infty]} \frac{1}{1+x}\, dm = \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{1}{1+x}\, dx \ge \sum_{n=1}^{\infty} \frac{1}{1+n} = \infty.$$

Ex. Let $f = \frac{1}{\sqrt{x}} \chi_{(0,1]}$ and write $f_n(x) = f(|x-n|)$.

Define $g = \sum \frac{1}{2^n} f_n$. Then $g \in L^1(\mathbb{R})$ but $g \notin L^2(\mathbb{R})$. For this consider

$$\int_{\mathbb{R}} g\, dm = \sum_{n=0}^{\infty} \frac{1}{2^n} \int f_n\, dm = \sum_{n=0}^{\infty} \frac{1}{2^n} \int_{\mathbb{R}} \frac{1}{\sqrt{|x-n|}} \chi_{(n,n+1]}(x)\, dm$$
$$= \sum \frac{1}{2^n} \int_{(0,1]} \frac{1}{\sqrt{x}}\, dm = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot 2 = 4.$$

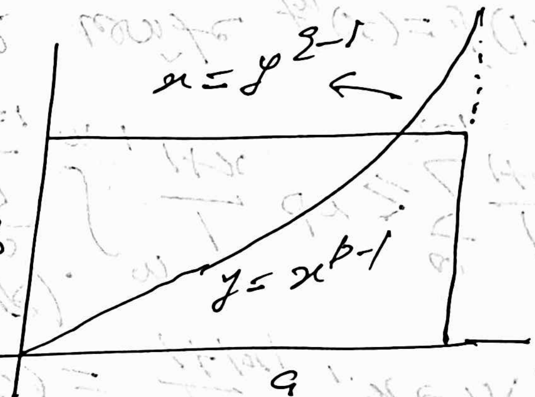
Now, $\int_R g^2\, dm = \sum \frac{1}{2^{2n}} \int_R |f_n|^2\, dm = \sum \frac{1}{2^{2n}} \int_0^1 \frac{1}{x}\, dm = \infty$

(Hint: use the fact that if $E_1 \cap E_2 = \emptyset$, then χ_{E_1} & χ_{E_2} are linearly independent)

For $1 \le p < \infty$, if we define
$$\|f\|_p := \left(\int |f|^p\right)^{1/p} < \infty,$$ then the space $L^p(X, S, \mu)$ is a normed linear space. If $\|f\|_p = 0 \iff f = 0$ a.e, and $\|cf\|_p = |c|\,\|f\|_p$. All we need to prove is the triangle inequality
$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

Young's inequality:

Let $1 < p, q < \infty$ & $\frac{1}{p} + \frac{1}{q} = 1$. If $a, b \ge 0$. Then
$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad \text{---}(*).$$



Proof: Let $y = x^{p-1}$. Then
$x = y^{q-1}$ $\left(\because \frac{1}{p} + \frac{1}{q} = 1\right)$.

and $ab \le \int_0^a x^{p-1}dx + \int_0^b y^{q-1}dy$

i.e $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

Note that equality in $(*)$ holds iff $a^p = b^q$ ($\Leftrightarrow a = b^{q-1}$). Consider $ab = \frac{a^p}{p} + \frac{b^q}{q}$.

Replace $a \to a^{1/p}$ & $b \to b^{1/q}$, $\frac{1}{p} = \alpha$. Then
$$a^\alpha b^{1-\alpha} = \alpha a + (1-\alpha) b$$

Put $\frac{a}{b} = t$, then $t \in (0, \infty)$ and
$$t^{\alpha} - \alpha t - (1-\alpha) = 0.$$ Define
$$f(t) = t^{\alpha} - \alpha t - (1-\alpha), \quad t \in (0, \infty).$$
Notice that $f(1) = 0$ & $f'(t) = \alpha(t^{\alpha-1} - 1) = 0$ iff $t = 1$. Hence f attains its maximum on $[0,1]$ at $t=1$. Because $f(t) \le f(1) = 0$, for $t \in (0,1]$. Hence $f(t) = 0$ iff $t = 1$ iff $a = b$.

Holder's inequality:
Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for $f \in L^p(X)$ and $g \in L^q(X)$, $fg \in L^1(X)$ and $\|fg\|_1 \le \|f\|_p \|g\|_q$.

Proof: We know that $ab \le \frac{a^p}{p} + \frac{b^q}{q}$. Let
$$a = \frac{|f|}{\|f\|_p}, \quad b = \frac{|g|}{\|g\|_q}. \text{ Then}$$
$$\int_X \frac{|fg|}{\|f\|_p \|g\|_q} \le \int_X \frac{|f|^p}{p\|f\|_p^p} + \int_X \frac{|g|^q}{q\|g\|_p^p}$$
$$\Rightarrow \|fg\|_1 \le \|f\|_p \|g\|_q.$$
Equality holds iff $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$.

Minkowski inequality:
Let $1 \le p < \infty$ & $f, g \in L^p(X)$. Then $f+g \in L^p(X)$ & $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Proof: For $p=1$, $f,g \in L^1(X)$ and we have seen $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$. (140)

Let $1<p<\infty$, and $\frac{1}{p}+\frac{1}{q}=1$. Then $p = q(p-1)$ and hence

$$\int |f+g|^{(p-1)q} = \int |f+g|^p < \infty.$$

$$\Rightarrow |f+g|^{p-1} \in L^q(X) \ \& \ |f| \in L^p(X).$$

By Holder's inequality, we get

$$\|f+g\|_p^p \le \int |f+g|^{p-1}(|f|+|g|)$$

$$= \int |f+g|^{p-1}|f| + \int |f+g|^{p-1}|g|$$

$$(1) \qquad \le \|f\|_p \, \|(f+g)^{p-1}\|_q + \|g\|_p \, \|(f+g)^{p-1}\|_q$$

But $\|(f+g)^{p-1}\|_q^q = \int |f+g|^{(p-1)q} = \int |f+g|^p$

$$= \|f+g\|_p^p \qquad -(2)$$

From (1) & (2), we get

$$\|f+g\|_p^{p(1-\frac{1}{q})} \le \|f\|_p + \|g\|_q$$

$$\Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_q.$$

Proposition: Let (X, S, μ) be a finite measure space. Let $1 \le p < q < \infty$. Then $L^q(X, S, \mu) \subseteq L^p(X, S, \mu)$.

Proof: $$\int_X |f|^p = \int_{\{x : |f(x)| < 1\}} |f|^p + \int_{\{x : |f(x)| \ge 1\}} |f|^p$$

$$\leq \mu(X) + \int_{\{x : |f(x)| \geq 1\}} |f|^q < \infty.$$

Further, let $r = \frac{q}{p}$. Then $r > 1$. Write $\frac{1}{r} + \frac{1}{s} = 1$.

Now, $|f|^p = |f|^{\frac{q}{r}} \in L^r(X)$. Hence

$$\int |f|^p = \int |f|^{\frac{q}{r}} \cdot 1 \leq \| |f|^q \|_r \| 1 \|_s.$$

$$\Rightarrow \|f\|_p \leq \left(\int |f|^q\right)^{\frac{1}{rp}} (\mu(X))^{\frac{1}{ps}}$$

$$= \|f\|_q \, (\mu(X))^{\frac{1}{p} - \frac{1}{q}}.$$

<u>Theorem</u>: For $1 \leq p < \infty$, the space $L^p(X, S, \mu)$ is complete. Moreover, if $f_n \to f$ in L^p, then $\exists$ a subsequence $f_{n_K}(x) \to f(x)$ pointwise a.e.

<u>Proof</u>: Let $\{f_n\}$ be a Cauchy seq^n in $L^p(X)$.

Then $\|f_{n_{j+1}} - f_{n_j}\|_p < \frac{1}{2^j}$, $\forall j \geq 1$ (Ex.)

write $f = f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j})$ —— (1)

and $g = |f_{n_1}| + \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}|$ —— (2)

Then $S_K(g) = |f_{n_1}| + \sum_{j=1}^{K} |f_{n_{j+1}} - f_{n_j}| \uparrow g$ p.w.

By Minkowski inequality

$$\|S_K(g)\|_p \leq \|f_{n_1}\|_p + \sum_{j=1}^{K} \frac{1}{2^j} \leq \|f_{n_1}\|_p + 1 < \infty.$$

i.e. $\int_X S_K(g)^p \uparrow$ & bounded above. Hence

By MCT, $\int_X g^p = \lim_{k\to\infty} \int S_k(g)^p < \infty$. Thus, (142)

$g \in L^p$. From (1) & (2), we get

$$|f| \le g \in L^p.$$

This implies, f is finite a.e. on X.

That is, $S_K(f) \xrightarrow[a.e]{p.w.} f \Rightarrow f_{n_K} \xrightarrow[a.e.]{p.w.} f$

Now, $|f_{n_K} - f|^p \to 0$ & we have

$$|f_{n_K} - f|^p \le 2^p (|f_{n_K}|^p + |f|^p)$$
$$= 2^p (|S_K(f)|^p + |f|^p)$$
$$\le 2^p (S_K(g)^p + g^p) \le 2^p . 2\, S_K(g)^p.$$

By DCT, $\lim \int |f_{n_K} - f|^p = 0$

i.e. $\|f_{n_K} - f\|_p \to 0$ & f_n is C.S.

in a n.l.s $L^p(X)$. Hence $f_n \to f$ $L^p(X)$.

The 2nd part of theorem will be followed by the fact that every conv. seq^n is C.S.

<u>Lemma</u>: The space of simple integrable functions are dense in $L^p(X, S, \mu)$, for $1 \le p < \infty$.

Proof: Let $S_p = \{ \varphi : X \xrightarrow{\text{ms'ble}} \mathbb{R} \text{ \& } \varphi \in L^p \}$.

For $f \in L^p$, f is measurable. Hence

$\exists$ a seq^n of m'ble simple functions φ_n s.t $\varphi_n \xrightarrow{p.w.} f$ & $|\varphi_n| \uparrow |f|$ p.w. That gives

$|\varphi_n|^p \le |f|^p \in L^1(X) \Rightarrow \varphi_n \in S_p$. (143)

Now, $|f-\varphi_n|^p \le 2^p(|f|^p+|\varphi_n|^p) < 2^{p+1}|f|^p \in L^1$.

By DCT, $\lim \int |f-\varphi_n|^p = \int \lim |f-\varphi_n|^p = 0$.

ie $\lim \|f-\varphi_n\|_p = 0$.

Thus, simple functions are dense in $L^p(X)$, $1 \le p < \infty$.

There are more class of functions which are dense in $L^p(X, S, \mu)$, $1 \le p < \infty$, if the space endowed with appropriate topology. One of them is the space of compactly supported continuous functions.

The support of a function f on a topological space is the closure of the set $\{x \in X : f(x) \ne 0\}$, and we denote

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \ne 0\}}.$$

If $\operatorname{supp}(f) \subseteq K$, and K is cpt, then we say f is compactly supported.

Ex. $f(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x|<1 \\ 0 & \text{o.w.} \end{cases}$

is a compactly supported function on $\mathbb{R}$ with $\operatorname{supp}(f) = \{x : |x| \le 1\}$.

In fact, given any compact set K in a locally compact Hausdorff space X, we can always construct a compactly supported continuous function. (44)

Let $C_c(X) = \{ f : X \xrightarrow{\text{cont}} \mathbb{R},\ \text{supp} f \subseteq K,\ K \text{ cpt} \}$.

<u>Urysohn's lemma</u>: Let K & O be compact and open sets in a locally cpt Hausdorff space X. If $K \subset O$, then $\exists\ f \in C_c(X)$ s.t. $f = 1$ on K, $f = 0$ on O^c & $0 \le f \le 1$.

For a proof of this result, I refer to Rudin (Real & Complex), page 39.

<u>Theorem</u>: $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R}, \mathcal{M}, m)$, if $1 \le p < \infty$.

<u>Proof</u>: Let $f \in L^p(\mathbb{R})$. Then $\exists$ a seqn φ_n of simple msble functions s.t. $|\varphi_n| \uparrow |f|$ p.w. In fact, $\forall\ \epsilon > 0$,

$\exists\ \varphi \in S_p$ s.t. $\|f - \varphi\|_p < \epsilon/2$ ——— (1)

Since $\varphi \in S_p \subseteq L^p$, we can write

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i},\quad m(E_i) < \infty,\ \text{if } \alpha_i \neq 0.$$

Since $m(E_i) < \infty$, for each $\epsilon > 0$, $\exists$ $K_i \subset E_i \subset O_i$ s.t.

$$m(O_i \setminus K_i) < \left(\frac{\epsilon}{2|\alpha_i| n}\right)^p \quad \text{---} (*)$$

Now, by Urysohn's Lemma, $\exists$ a function $g_i \in C_c(\mathbb{R})$ s.t. $g_i|_{K_i} = 1$, & $g_i|_{O_i^c} = 0$.

Hence $\int_{\mathbb{R}} |\chi_{E_i} - g_i|^p = \int_{O_i} |\chi_{E_i} - g_i|^p = \int_{O_i \setminus K_i} |\chi_{E_i} - g_i|^p$

$$\leq m(O_i \setminus K_i) < \left(\frac{\epsilon}{2|\alpha_i|}\right)^p \frac{1}{n^p}$$

i.e. $\|\chi_{E_i} - g_i\|_p < \frac{\epsilon}{2|\alpha_i| n}$

Let $g = \sum_{i=1}^{n} \alpha_i g_i$. Then $\varphi - g = \sum_{i=1}^{n} \alpha_i (\chi_{E_i} - g_i)$.

Hence, $\|\varphi - g\|_p \leq \sum_{i=1}^{n} |\alpha_i| \, \|\chi_{E_i} - g_i\|_p < \epsilon/2 \quad \text{---} (2)$

From (1) & (2),

$$\|g - f\|_p \leq \|g - \varphi\|_p + \|\varphi - f\|_p < \epsilon.$$

Notice that if $m(E) < \infty$, then $\exists$ $K \subset E \subset O$ s.t. $m(O \setminus E) \leq m(O \setminus K) < \epsilon$, for $\epsilon > 0$.

Then $\|\chi_O \setminus \chi_E\|_p < \epsilon^{1/p}$. But $O = \bigcup_{n=1}^{\infty} I_n$, & $m(O \setminus \bigcup_{n=1}^{K} I_n) < \epsilon$, for $K > K_0$.

Let $\psi_K = \sum_{n=1}^{K} \chi_{I_n}$. Then $\|\chi_O \setminus \psi_K\|_p < \epsilon^{1/p}$.

This shows that $L^p(\mathbb{R})$ can be constructed over $\{\chi_{I_n} : I_n$ - open & bounded$\}$. Hence $L^p(\mathbb{R})$ is a seperable normed linear space. 146

That is, for $f \in L^p(\mathbb{R})$ and $\epsilon > 0$, $\exists$

$$\psi = \sum_{i=1}^{n} \alpha_i \chi_{I_i}, \quad |\alpha_i| < \infty, \quad m(I_i) < \infty$$

Such that $\quad \|f - \psi\|_p < \epsilon$.

Function ψ is called step function.

Now, if $f \in L^p[a,b]$, then $\|\psi - f\|_p < \epsilon$ & $\psi \in R[a,b]$. Hence $\overline{R[a,b]} = L^p[a,b]$, if $1 \le p < \infty$.

<u>Theorem</u>: Let (X, S, μ) be a regular measure space on a LCH X. Then $\overline{C_c(X)} = L^p(X)$.

Note that μ is said to be regular if

(i) $\mu(K) < \infty$, $\forall$ cpt set $K \subset X$.

(ii) $\mu(E) = \sup\limits_{K \subset E} \mu(K) = \inf\limits_{O \supset E} \mu(O)$.

Example: $(\mathbb{R}, \mathcal{M}, m)$ is regular.

<u>Functions vanishing at ∞.</u>

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be vanishing at ∞ if $\lim\limits_{|x| \to \infty} f(x) = 0$. If f is continuous, then f is bounded for $\lim\limits_{|x| \to \infty} f(x) = 0$.

In fact for $\epsilon>0$, $\exists\, \delta>0$ s.t

$$|f(x)|<\epsilon, \quad \forall\, x: |x|\geq \frac{1}{\delta}.$$

Let $C_0(\mathbb{R}) = \{ f: \mathbb{R} \xrightarrow{\text{cont}} \mathbb{R} \text{ \& } \lim_{|x|\to\infty} f(x) = 0 \}$.

Then $(C_0(\mathbb{R}), \|\cdot\|_u)$ is a complete n.l.s, Here $\|f\|_u := \sup_{x\in\mathbb{R}} |f(x)|$. If f_n is a C.S. in $C_0(\mathbb{R})$, then for $\epsilon>0$, $\exists\ n_0\in\mathbb{N}$ s.t

$$\sup_{x\in\mathbb{R}} |f_n(x) - f_m(x)| < \epsilon, \quad \forall\, n, m \geq n_0.$$

$$\text{i.e. } |f_n(x) - f_m(x)| < \epsilon, \quad \forall\, n, m \geq n_0, \ \forall\, x\in\mathbb{R}.$$

Let $\lim_{n\to\infty} f_n(x) = f(x)$ ($\because f_n(x)$ is C.S in $\mathbb{R}$).

Then $|f(x) - f_m(x)| < \epsilon$, $\forall\, m\geq n_0$, $\forall\, x\in\mathbb{R}$

Since $f_m \in C_0(\mathbb{R})$, letting $|x|\to\infty$, we get

$$\lim_{|x|\to\infty} |f(x)| \leq \epsilon, \quad \forall\, \epsilon>0.$$

$$\text{i.e } \lim_{|x|\to\infty} |f(x)| = 0.$$

Further, $\overline{C_c(\mathbb{R})} = C_0(\mathbb{R})$. For this, let $f\in C_0(\mathbb{R})$. Then for $\epsilon>0$, $\exists$ cpt set $K\subset\mathbb{R}$ s.t

$$|f(x)|<\epsilon, \quad \forall\, x\in K^c.$$

By Urysohn's lemma, $\exists$ open set $O\supset K$ and $g\in C_c(\mathbb{R})$ s.t $g=1$ on K & $g=0$ on O^c. Let $h = fg$. Then $h\in C_c(\mathbb{R})$

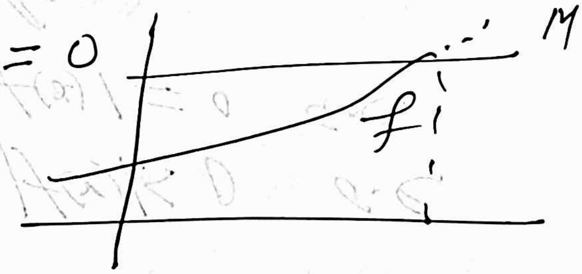
and $|f(x)-h(x)| = |f(x)(1-g(x))| \le |f(x)| < \epsilon$
($\because$ $0 \le g(x) \le 1$ $\forall\, x \in X$). Thus,

$$\|f-h\|_u < \epsilon.$$

$\underline{L^\infty(X,S,\mu)}$: A measurable function f on X is said to be essentially bounded on X w.r.t. μ if $\exists\, M \ge 0$ such that

$$\mu\{x \in X : |f(x)| > M\} = 0$$

(or $|f(x)| \le M$ a.e.)



Notice that if $|f(x)| > M+1$ then $|f(x)| > M$. Hence

$$\mu\{x \in X : |f(x)| > M+1\} = 0.$$

Thus, we need to minimize M for f.

Denote
$$\|f\|_\infty := \inf\{M : |f(x)| \le M, \text{ a.e.}\}$$
$$= \operatorname{ess.sup}_{x \in X} |f(x)|.$$

If no such M exists for f, then we say $\|f\|_\infty = \infty$, by the convention that $\inf \emptyset = \infty$.

Now, for $n \in \mathbb{N}$, $\exists\, M_n > 0$ s.t.
$$\|f\|_\infty + \frac{1}{n} > M_n.$$

Then $\{x \in X : |f(x)| > \|f\|_\infty\} = \bigcup \{x \in X : |f(x)| > \|f\|_\infty + \frac{1}{n}\}$
$\subset \bigcup \{x \in X : |f(x)| > M_n\}$

Since $\mu\{x : |f(x)| > M_n\} = 0$, it follows that
$$\mu\{x \in X : |f(x)| > \|f\|_\infty\} = 0.$$

Hence, $|f(x)| \leq \|f\|_\infty$ a.e. on X. It is clear that $\|f\|_\infty \leq \sup_{x\in X} |f(x)|$, however both of them need not be same. (149)

ex. Let $f(x) = \chi_{\mathbb{Q}}(x)$, $\mathbb{Q} \subset \mathbb{R}$, the set of rationals. Then $\|f\|_\infty = 0 < \sup_{x\in\mathbb{R}} |f(x)| = 1$.

Consider $f(x) = \frac{1}{\sqrt{x}}$, $x > 0$, then $f \notin L^\infty(0,\infty)$.

Since $\frac{1}{\sqrt{x}} < M \Rightarrow 0 < \frac{1}{M^2} < x$. However, $\frac{1}{\sqrt{x}} \in L^1(0,\infty)$. Thus, in general $L^\infty(X) \not\subset L^p(X)$, $1 \leq p < \infty$.

If $\mu(X) < \infty$, then $L^\infty(X,S,\mu) \subset L^p(X,S,\mu)$.

Let $f \in L^\infty$, then

$$\int_X |f|^p \leq \mu(X) \, \|f\|_\infty^p$$

$$\|f\|_p \leq (\mu(X))^{1/p} \|f\|_\infty.$$

Notice that $\|f\|_\infty = 0 \Leftrightarrow |f(x)| \leq 0$ a.e.

$\Leftrightarrow |f(x)| = 0$ a.e.

Also, $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Hence $L^\infty(X,S,\mu)$ is a normed linear space.

Remark: If $0 < \alpha < \|f\|_\infty$, then $\mu\{x\in X : |f(x)| > \alpha\} \neq 0$.

<u>Theorem:</u> $L^\infty(X,S,\mu)$ is a complete n.l.s.

Proof: Let $\{f_n\}$ be a c.s. in L^∞.

Then for $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t
$$\|f_n - f_m\|_\infty < \epsilon, \ \forall\ n, m \geq N. \quad (150)$$

But then $|f_n(x) - f_m(x)| < \epsilon$, a.e. on X, $\forall$ $m, n \geq N$.

Then $|f_n(x) - f_m(x)| < \epsilon$, $\forall$ $x \notin E_N$, $\forall$ $n, m \geq N$,

where $E_N = \bigcup_{m,n \geq N} E_{m,n}$. $E_{m,n} = \{x : |f_n(x) - f_m(x)| \geq \epsilon\}$.

But $\mu(E_N) = 0$. Thus, for each $x \in E_N^c$, $\{f_n(x)\}$ is C.S. in $\mathbb{R}$. Let
$$f(x) = \lim f_n(x), \quad x \in E_N^c.$$

Then $|f_m(x) - f(x)| \leq \epsilon$, $\forall$ $x \in E_N^c$, $\forall$ $m \geq N$.

& $\|f_m - f\|_\infty \leq \epsilon$, $\forall$ $m \geq N$.

$\|f\|_\infty \leq \|f_N - f\|_\infty + \|f_N\|_\infty < \epsilon + \|f_N\|_\infty < \infty$.

Hence $f \in L^\infty$ & $f_m \to f$ in L^∞.

Theorem: Let $S = \{\varphi : (X, S, \mu) \xrightarrow{\text{msble}} \bar{\mathbb{R}}$, simple function & $\varphi \in L^\infty(X, S, \mu)\}$.

Then S is dense in $L^\infty(X, S, \mu)$.

Proof: Let $f \in L^\infty$, then $\exists$ a seq^n φ_n of simple mssble functions s.t
$$\varphi_n \xrightarrow{p.w.} f \ \ \& \ |\varphi_n| \uparrow |f| \ \text{p.w.}$$
$$\Rightarrow |\varphi_n| \leq |f| \in L^\infty \Rightarrow \varphi_n \in L^\infty.$$

We know that $|f(x)| \leq \|f\|_\infty$ a.e. on X.

Let $E = \{x : |f(x)| \leq \|f\|_\infty\}$. Then $\mu(E^c) = 0$, & φ_n converges uniformly to f on E.

Hence for $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t. (151)

$$|\varphi_n(x) - f(x)| < \epsilon, \quad \forall\, n \geq n_0, \ \forall\, x \in E.$$

$$\Rightarrow \|(\varphi_n - f)|_E\|_\infty \leq \epsilon, \quad \forall\, n \geq n_0.$$

$$\Rightarrow \|\varphi_n - f\|_\infty \leq \epsilon, \quad \forall\, n \geq n_0$$

$(\because \|g\|_\infty = \|g|_{X \setminus E}\|_\infty$, if $\mu(E) = 0)$.

Ex. If $\mu(X) < \infty$, then $\lim\limits_{n\to\infty} \|f\|_p = \|f\|_\infty$.

We know that $\|f\|_p \leq \|f\|_\infty (\mu(X))^{1/p}$, we get

$$\lim \|f\|_p \leq \|f\|_\infty \lim (\mu(X))^{1/p} = \|f\|_\infty. \quad \text{--- (1)}$$

Now, for $\epsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\mu\{x \in X : |f(x)| > \|f\|_\infty - \epsilon\} > \delta.$$

Let $E = \{x : |f(x)| > \|f\|_\infty - \epsilon\}$. Then

$$\int_X |f|^p d\mu \geq \int_E |f|^p d\mu \geq (\|f\|_\infty - \epsilon)^p \mu(E)$$

$$\Rightarrow \|f\|_p \geq (\|f\|_\infty - \epsilon)\, \mu(E)^{1/p}$$

$$\Rightarrow \underline{\lim} \|f\|_p \geq (\|f\|_\infty - \epsilon) \underline{\lim} (\mu(E))^{1/p}$$

or $\underline{\lim} \|f\|_p \geq (\|f\|_\infty - \epsilon) \cdot 1$, $\forall\, \epsilon > 0$. --- (2)

From (1) & (2) $\underline{\lim} \|f\|_p \geq \|f\|_\infty \geq \overline{\lim} \|f\|_p$.

Theorem: $L^\infty(\mathbb{R}, \mathcal{M}, m)$ is not separable.

Proof: Let $f_t = \chi_{[0,t)}$, $t \in (0,1)$. Then

$\|f_s - f_t\|_\infty = 1$, if $s < t$, and

$S = \{B_{1/2}(f_t) : t \in (0,1)\}$ is an uncountable family of disjoint open balls in $L^\infty(\mathbb{R})$.

If A is a countable dense set in $L^\infty(\mathbb{R})$, then every open ball in $L^\infty(\mathbb{R})$ has to intersect A. But that is not the case, because S itself an uncountable family of disjoint balls. That is, $\exists$ $t_0 \in (0,1)$ s.t. $B_{1/2}(f_{t_0}) \cap A = \Phi$.

Hence, $L^\infty(\mathbb{R}, \mathcal{M}, m)$ is unable to carry a countable dense set in itself.

Dual of L^p-space:

Let $(X, \|\cdot\|)$ be a n.l.s. on $\mathbb{R}$ (or $\mathbb{C}$). A linear map $T : X \longrightarrow \mathbb{R}$ is said to be bounded if $\exists$ $M > 0$ s.t. $|Tx| \leq M\|x\|$, $\forall x \in X$.

Then $\|T\| := \inf\{M : |Tx| \leq M\|x\|\}$

$$= \sup\left\{\frac{|Tx|}{\|x\|} : x \in X, x \neq 0\right\}$$

$$= \sup_{\|x\| \leq 1} |Tx| = \sup_{\|x\| = 1} |Tx|.$$

Let $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq \infty$. For $g \in L^q(X, \mathcal{S}, \mu)$, Write $T_g(f) = \int_X fg \, d\mu$, for $f \in L^p(X, S, \mu)$. (153)

Then $T_g : L^p(X, S, \mu) \to \mathbb{C}$ is a bounded linear functional. For this, by Holder's inequality we get $|T_g(f)| \leq \|f\|_p \|g\|_q$. If $\|f\|_p = 1$, then by taking supremum of both the sides, we get $\|T_g\| \leq \|g\|_q$ —— (1)

Thus, $T_g \in (L^p)^* = \{ T : L^p \xrightarrow[\text{linear}]{\text{bdd}} \mathbb{C} \,(\text{or } \mathbb{R}) \}$, $\forall g \in L^q$.

That is, $L^q \hookrightarrow (L^p)^*$.

Next, we show that inequality in (1) is essentially equality.

Case (i) Let $p = 1$ & $q = \infty$. Suppose X is a σ-finite measure space. Then $X = \dot{\bigcup}_{n=1}^{\infty} E_n$, & $\mu(E_n) < \infty$. For $\epsilon > 0$, we must have

$$\mu\{x \in X : |g(x)| > \|g\|_\infty - \epsilon\} > 0.$$

Let $E = \{x \in X : |g(x)| > \|g\|_\infty - \epsilon\}$. Then

$E = \dot{\bigcup}_{n=1}^{\infty} (E_n \cap E)$. Since $\mu(E) > 0$,

$\exists\, n_0 \in \mathbb{N}$ s.t. $0 < \mu(E_{n_0} \cap E) < \infty$.

Hence, w.l.o.g. we can assume $0 < \mu(E) < \infty$.

(125)

Consider $f_0 = \frac{\chi_E}{\mu(E)} \operatorname{Sign} g$. Then $\|f\|_1 = 1$.

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Now, $T_g(f_0) = \frac{1}{\mu(E)} \int_E g \operatorname{Sign}(g)\, d\mu$

$= \frac{1}{\mu(E)} \int_E |g|\, d\mu \geq \|g\|_\infty - \epsilon, \ \forall \epsilon > 0.$

Hence $\|g\|_\infty \leq T_g(f_0) \leq \|T_g\| \leq \|g\|_\infty$.

i.e. $\|g\|_\infty = \|T_g\|$.

Case II If $p = \infty$, $q = 1$ (no need to take X to be σ-finite).

Let $f_0 = \operatorname{Sign} g$. Then $\|f_0\|_\infty = 1$, and

$T_g(f_0) = \int g \operatorname{Sign} g\, d\mu = \int |g|\, d\mu = \|g\|_1$.

Hence, $\|T_g\| = \|g\|_1$.

Case IV $1 < p < \infty$ (no need to consider X as σ-finite).

Let $f_0 = \frac{|g|^{q-1}}{\|g\|_q^{q-1}} \operatorname{Sign} g$, $g \in L^q$.

Then $f_0 \in L^p$ & $\|f_0\|_p = 1$ ($\because p(q-1) = q$).

Now, $T_g(f_0) = \int \frac{|g|^q}{\|g\|_q^{q-1}}\, d\mu = \|g\|_q$.

Hence $\|T_g\| = \|g\|_q$.

Thus, we have shown that $L^q \subseteq (L^p)^*$ with $T_g(f) = \int fg$ & $\|T_g\| = \|g\|$.

However, for $1 < p < \infty$, we prove later that

$$L^q(X, S, \mu) \cong (L^p(X, S, \mu))^*$$ for any

measure space (X, S, μ). On the other hand if (X, S, μ) is σ-finite, then

$$L^\infty(X, S, \mu) \cong (L^1(X, S, \mu))^*.$$

Remark: $L^1 \subsetneq (L^\infty)^*$.

We know that $\overline{S(\mathbb{R})} = L^\infty(\mathbb{R})$, where $S(\mathbb{R})$ is the space of all essentially bounded simple functions.

Let $T : S(\mathbb{R}) \to \mathbb{C}$, be defined by

$T(\varphi) = \varphi(0)$. Then $\|T\| = 1$.

By Hahn-Banach theorem, T can be extended to $L^\infty(\mathbb{R})$.

Suppose $(L^\infty)^* \cong L^1$. Then $\exists\, f_0 \in L^1$ s.t. $T = T_{f_0}$ & $\|T\| = \|f_0\|_1 = 1$.

For $I \subset \mathbb{R} \setminus \{0\}$, I bdd,

$$0 = T(\chi_I) = \int f_0 \chi_I = \int_I f_0, \ \forall I.$$

Hence, $f_0 = 0$ a.e., which contradicts that $(L^\infty)^* \cong L^1$.

Lemma: Let $1 \le p \le \infty$ & $\frac{1}{p}+\frac{1}{q}=1$. Suppose g is a measurable function on a σ-finite measure space (X, S, μ) such that (156)

(*) $M_g = \sup\{|\int_X fg| : f \text{ simple } \& \ \|f\|_p = 1\} < \infty$.

Then $g \in L^q$ and $M_g = \|g\|_q$.

Proof: Since g is measurable, $\exists$ a seq^n of simple functions $\varphi_n \to g$ p.w. & $|\varphi_n| \le |g|$. Given that X is σ-finite, $\exists$ an increasing seq^n E_n of finite measure s.t. $E_n \uparrow X$.

Hence, $g_n = \varphi_n \chi_{E_n} \to g$ & $|g_n| \le |g|$.

Since $|\int_X fg| \le M_g$, $\forall$ f simple & $\|f\|_p = 1$, by replacing f with $|f| \operatorname{sign} g$, we get

$\int_X |fg| \le M_g$, whenever, f simple, $\|f\|_p = 1$.

Then $\int_X |fg_n| \le \int_X |fg| \le M_g$ - - - (1)

Case I $q < \infty$: For $f = \chi_{E_n} / \mu(E_n)^{1/p}$, $\|f\|_p = 1$.

Hence, $\int_{E_n} |g_n| \le M_g \, \mu(E_n)^{1/p} < \infty$. That is, $g_n \in L^1(X, S, \mu)$ and g_n is simple. Thus, $g_n \in L^q(X, S, \mu)$, (because every simple L^1 function is in any L^q).

$\left(\because \int |g_n|^q = \int \sum_{i=1}^{n} |k_i|^q \chi_{E_i} = \sum |k_i|^q \mu(E_i) < \infty\right)$. (157)

Further, consider $f_n = \dfrac{g_n^{\,q-1}}{\|g_n\|_q^{\,q-1}}$, then $\|f_n\|_p = 1$ and f_n is simple. Hence, from (1), we get

$$\|g_n\|_q \le M_g .$$

By Fatou's lemma,

$$\int_X |g|^q \le \underline{\lim} \int_X |g_n|^q \le M_g^{\,q} \Rightarrow \|g\|_q \le M_g < \infty.$$

that is $g \in L^q$. By Holder's inequality

$$\left|\int_X fg\right| \le \int_X |fg| \le \|f\|_p \|g\|_q = \|g\|_q, \text{ if } \|f\|_p = 1.$$

$$\Rightarrow M_g = \sup \left|\int_X fg\right| \le \|g\|_q .$$

<u>Case $q=\infty$</u>: In this case $p=1$, and we can take $f_n = \dfrac{g_n}{\|g_n\|_\infty}$, $\|f_n\|_1 =$

Consider the set $E = \{x \in X : |g(x)| \ge M_g + \epsilon\}$ having +ve measure, for each $\epsilon > 0$.

Since (X, S, μ) is σ-finite, w.l.g. we can assume that $0 < \mu(E) < \infty$. Setting

$f = \dfrac{\chi_E \operatorname{sign} g}{\mu(E)}$. Then $\|f\|_1 = 1$, and

$$M \ge \left|\int_X fg\right| = \int_X fg = \int_E \frac{\chi_E |g|}{\mu(E)} \ge M_g + \epsilon,$$ is

impossible. Hence $|g(x)| \le M_g$ a.e.

$\Rightarrow$ $\|g\|_\infty \le M_g$. Again by Holder's inequality, $|\int fg| \le \|f\|_1 \|g\|_\infty = \|g\|_\infty$, if $\|f\|_1 = 1$.

$\Rightarrow$ $M_g \le \|g\|_\infty$.

(158)

Remark: From the proof, it is clear that, to prove $g \in L^q$, (or $M_g = \|g\|_q$), it is enough to take supremum in (*) on those L^p-simple functions which vanish outside set of finite measure.

Signed measure:

A set function ν on a measurable space (X, S) is said to be signed measure if (i) $\nu(\phi) = 0$ (definiteness)

(ii) ν assume at most one of the value $+\infty$ or $-\infty$

(i.e $\nu(E) \in (-\infty, \infty]$ (or $[-\infty, \infty)$)).

(iii) $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum \nu(E_i)$, for every disjoint countable family $\{E_i\}_{i=1}^{\infty} \subset S$.

(i.e the series in RHS converges absolutely while $\nu(\bigcup E_i) < \infty$.

Ex. If μ_1 & μ_2 be two measures on (X, S) with one of them is finite, then

$\nu = \mu_1 - \mu_2$ is a signed measure.

Ex. If f is a measurable function on (X, S, μ) s.t either $\int f^+ d\mu$ & $\int f^- d\mu$ is finite, then

$$\nu(E) = \int_E f^+ - \int_E f^-$$ is a signed measure on (X, S).

However, these two examples of signed measure are not isolated, rather, any signed measure can be expressed in either of them. We see it later.

<u>Lemma</u>: Let ν be a signed measure on a measure space (X, S).

(i) If $\{E_i\}_{i=1}^{\infty}$ $\uparrow$ family in S, then
$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim \nu(E_n).$$

(ii) If $\{E_i\}_{i=1}^{\infty}$ is $\downarrow$ & $\nu(E_1)$ is finite,
then $$\nu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim \nu(E_i).$$

Proof is same as to the case of positive measure, and we omit here.

Def^n: Let ν be a signed measure on (X, S). A set $E \in S$ is said to be +ive set (resp. -ive set, null set) if $\nu(F) > 0$ (or $\nu(F) < 0$, $\nu(F) = 0$) for each $F \subseteq E$ and $F \in S$.

Ex. If $\nu(E) = \int_E f \, d\mu$ where μ is a +ve measure, and at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite, then E is a positive, negative, null set if $f > 0$, $f < 0$ or $f = 0$ a.e. μ on E.

<u>Lemma</u>: Union of any countable family of positive set is a positive set.

<u>Proof</u>: Let $P_1, P_2, \dots$ be +ve sets for ν. Write $Q_n = P_n \setminus \bigcup_{i=1}^{n-1} P_i$; $n \geq 2$. Then $Q_n \subset P_n$ and $\bigcup Q_n = \bigcup P_n$.

If $E \subset \bigcup_{i=1}^{\infty} P_i$, then

$$\nu(E) = \nu\left(\bigcup (Q_n \cap E)\right)$$
$$= \sum \nu(Q_n \cap E) > 0.$$

Next, we see that any set X can be written as disjoint union of +ve & -ve sets.

Hahn decomposition thm:

Let ν be a signed measure on (X, S). Then $\exists$ a +ve set P and negative set N s.t. $X = P \dot{\cup} N$.

Proof: Without loss of generality, we can assume that ν does not take value $+\infty$ (otherwise, we consider $-\nu$).

Let $m = \sup\{\nu(E) : E$ is +ve set$\}$.
Since class of all +ve sets is non-empty as it contains empty set. Hence
$-\infty < m < \infty$.
Moreover, $\exists$ a seq^n P_i of +ve sets s.t. $\nu(P_i) \to m$.
Let $P = \bigcup_{i=1}^{\infty} P_i$. Then P is a +ve set, and $P_i \subset P$. Hence
$$\nu(P_i) \le \nu(P) \le m.$$
Also, $\nu(P) = \nu((P \setminus P_i) \dot{\cup} P_i) \ge \nu(P_i) \to m$.
$$\Rightarrow \nu(P) = m.$$

Let $N = X \setminus P$. We show that N is a negative set.

Notice that N cannot contain any nonnull positive sets. Indeed, if $E \subset N$ is a +ve set and $\nu(E) > 0$, then $E \dot\cup P$ is a +ve set, and

$$\nu(E \dot\cup P) = \nu(E) + \nu(P) > m,$$ a contradiction.

On the other hand, if $A \subset N$ and $\nu(A) > 0$, then $\exists\ B \subset A$ with $\nu(B) > \nu(A)$.

This is possible, because A cannot be a +ve set, and $\exists\ C \subset A$ with $\nu(C) < 0$.

Let $B = A \setminus C$, Then

$$\Rightarrow \nu(B) = \nu(A) - \nu(C) > \nu(A).$$

On contrary, suppose N is not a +ve set. Then we can find least +ve integer n_1 s.t

$$(*) \quad \frac{1}{n_1} = \max\left\{\frac{1}{n} : n \in \mathbb{N}, \exists\ B_1 \subset N, B_1 \in \mathcal{S} \text{ with } \nu(B_1) > \frac{1}{n}\right\}.$$

That is, n_1 is the least +ve integer such that $\exists\ B_1 \subset N$ & $\nu(B_1) > \frac{1}{n_1}$

(i.e $\nexists\ B \subset N$ s.t $\nu(B) \leq \frac{1}{n_1 - 1}$).

But B_1 cannot be a +ve set, hence $\exists$ least +ve integer n_2 and $B_2 \subset B_1$ s.t. $\nu(B_2) > \nu(B_1) + \frac{1}{n_2}$ (163)

By induction, $\exists\, B_i \subset B_{i-1}$ s.t.

$$\nu(B_i) > \nu(B_{i-1}) + \frac{1}{n_i} \quad \forall\, i > 2.$$

Let $B = \bigcap_{i=1}^{\infty} B_i$. Then

$$\infty > \nu(B) = \lim \nu(B_i) > \sum_{i=1}^{\infty} \frac{1}{n_i} > 0.$$

(because $\nu(B_i) > \nu(B_{i-2}) + \frac{1}{n_{i-1}} + \frac{1}{n_i}$ etc)

$\Rightarrow n_j \to \infty$ is possible

(i.e. the process is endless)

Notice that $0 < \nu(B) < \infty$. But B cannot be a +ve set. Hence $\exists\, C \subset B$ s.t. $\nu(C) > \nu(B)$.

But then we can find a large n_i s.t. $\nu(C) > \nu(B) + \frac{1}{n_{i-1}}$

This contradicts the construction of n_i (∵ n_i was least as defined by (*)).

Remark: If P' & N' is another decom-

-position of X. Then $P \setminus P' \subset P$ and $P \setminus P' \subset N \Rightarrow P \setminus P'$ is both +ve & -ve, hence $P \setminus P'$ is null set. Similarly $N - N'$ is null set. Thus

$$P \Delta P' = N \Delta N' = \text{null set}.$$

$X = P \cup N$ is known as Hahn decomposition for ν. It is not unique (upto ν-null set can be transfered from P to N or from N to P). However, it leads to a canonical decomposition of ν as the difference of two positive measures.

For this, we need the following concept:

Def^n: Two signed measures μ & ν on (X, S) are said to be mutually singular (or ν is singular w.r.t μ) if $\exists\ E, F \in S$ s.t. $E \cap F = \emptyset$, and $E \cup F = X$, E is null for μ and F is null for ν, and we write $\mu \perp \nu$.

Next, we decompose signed measure into two +ve measures.

Jordan decomposition thm

If ν is a signed measure, then $\exists !$ +ve measures ν^+ and ν^- s.t.

$\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof: let $X = P \cup N$ be a Hahn decomposition of ν, and let $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. Then

$\nu(E) = \nu(E \cap X) = \nu(E \cap P) + \nu(E \cap N)$

$\Rightarrow \nu(E) = \nu^+(E) - \nu^-(E)$.

obviously, $\nu^+ \perp \nu^-$.

If $\nu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$, let $E, F \in S$ s.t. $E \cap F = \Phi$, $E \cup F = X$, and $\mu^+(F) = 0$, $\mu^-(E) = 0$.

Then $X = E \cup F$ is another Hahn decomposition of ν. Hence $E \Delta P$ is a null set.

Now, for $A \in S$,

$\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P)$

$\Rightarrow \mu^+(A) = \nu^+(A) \Rightarrow \mu^+ = \nu^+$

Similarly, $\mu^- = \nu^-$.

The measures ν^+ and ν^- are called (166)
+ve and −ve variation of ν resp.
This is similar to function of bounded variation as difference of two increasing functions.

Also. $|\nu| = \nu^+ + \nu^-$ is called total variation of ν.

Remark: (i) If ν does not take value $+\infty$, then $\mu^+(X) = \nu(P) < \infty$. In particular, if the range of ν is contained in $\mathbb{R}$, then ν is bounded.

(ii) $\nu(E) = \int_E f \, d\mu$, where $\mu = |\nu|$,
$f = \chi_P - \chi_N$, $X = P \cup N$, a Hahn decomp. for ν.

We write
$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-) = L^1(|\nu|),$$
and for $f \in L^1(\nu)$,
$$\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-.$$
Note that ν is called finite (or σ-finite) if $|\nu|$ is finite (or σ-finite).

Ex. (i) $E \in S$ is null set for ν iff $|\nu|(E) = 0$.

(ii) $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ & $\nu^- \perp \mu$.

Ex. If $f \in L^1(\nu)$, then

$$\left|\int_X f\,d\nu\right| \leq \int |f|\,d|\nu|.$$

Ex. $E \in M$, then

$$|\nu|(E) = \sup\left\{\left|\int_E f\,d\nu\right| : |f| \leq 1\right\} = \alpha \text{ (say)}.$$

$$\left|\int_E f\,d\nu\right| \leq \int_E |f|\,d|\nu| \leq |\nu|(E) \quad (\because |f| \leq 1)$$

$$\Rightarrow \quad \alpha \leq |\nu|(E).$$

On the other hand, let $f_0 = \chi_P - \chi_N$;

$P \cup N = X$ is a Hahn decomposition.

$$\alpha \geq \left|\int_E f_0\,d\nu\right| = \left|\int_E f_0\,d\nu^+ - \int_E f_0\,d\nu^-\right|$$

$$= \nu^+(E \cap P) + \nu^-(E \cap N)$$

$$= |\nu|(E).$$

$$\Rightarrow \alpha = |\nu|(E).$$

Ex. (i) $\nu^+(E) = \sup\{\nu(F) : F \in S, F \subset E\}$

(ii) $\nu^-(E) = -\inf\{\nu(F) : F \in S, F \subset E\}$

(iii)

$$|\nu|(E) = \sup\left\{\sum_{i=1}^{n} |\nu(E_i)| : \bigcup_{i=1}^{n} E_i = E\right\} = \alpha$$

E_i's are disjoint sets.

$$\sum_{i=1}^{n} |\nu(E_i)| \le \sum_{i=1}^{n} (\nu^+(E_i) + \nu^-(E_i))$$ 168

$$= \nu^+(E) + \nu^-(E)$$

$$= |\nu|(E).$$

$\Rightarrow \quad \alpha \le |\nu|(E).$

On the other hand let $E_1 \cup E_2 = E$ be a Hahn decomposition of E. Then

$$\alpha \ge |\nu(E_1)| + |\nu(E_2)| = \nu^+(E_1) + \nu^-(E_2)$$

$$= \nu^+(E_1 \cup E_2) + \nu^-(E_1 \cup E_2)$$

$$= |\nu|(E)$$

$\Rightarrow \quad \alpha = |\nu|(E).$

Def^n: Let ν be a signed measure and μ be a +ve measure on (X, S). Then ν is called abs. cont. w.r.t. μ $(\nu \ll \mu)$ if $\mu(E) = 0 \Rightarrow \nu(E) = 0$.

It is easy to show that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Notice that abs. cont is antithesis of singularity.

Ex. If $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

Since $\nu \perp \mu$, $\exists E, F \in S$ s.t. $X = E \cup F$, and $\mu(E) = |\nu|(F) = 0$.
Also, $\nu \ll \mu$ & $|\mu|(E) = 0 \Rightarrow |\nu|(E) = 0$. 169
Hence $\nu = 0$.

The term abs. cont becomes more familiar for finite signed measure.

<u>Theorem</u>: Let ν be a finite signed measure and μ be a +ve measure on (X, S). Then $\nu \ll \mu$ iff $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow |\nu(E)| < \epsilon$.

Proof: Since $\nu \ll \mu$ iff $|\nu| \ll \mu$ and $|\nu(E)| \leq |\nu|(E)$, it is sufficient to assume that $\nu = |\nu|$ is +ve measure. Suppose $\nu \ll \mu$ and ϵ-δ condition fails. Then $\exists \epsilon > 0$ s.t. for all $n \in \mathbb{N}$, $\exists E_n \in S$ s.t. $\mu(E_n) < 2^{-n}$ & $\nu(E_n) \geq \epsilon$. Let $F_k = \bigcup_{n=k}^{\infty} E_n$ and $F = \bigcap_{k=1}^{\infty} F_k$. Then $\nu(F_k) \geq \epsilon$, $\forall k \geq 1$, and ν is finite,

$\nu(F) = \lim \nu(F_k) \geq \epsilon$, which a contradiction. to the fact that

$$\mu(F) \leq \mu(F_k) < \sum_{n=k}^{\infty} 2^{-n} = 2^{1-k} \to 0.$$

other implication follows easily.

Cor: If $f \in L^1(\nu)$, then for $\epsilon > 0$, $\exists\, \delta > 0$. st $|\nu|(E) < \delta \Rightarrow \left|\int_E f\, d\nu\right| < \epsilon$.

ex- let μ be a measure and

$$\nu(E) = \int_E f\, d\mu = \int_E f^+ d\mu - \int_E f^- d\mu,$$

where at least one of $\int_E f^+ d\mu$ or $\int_E f^- d\mu$ is finite, then $\nu << \mu$.

And ν is finite iff $f \in L^1(\mu)$.

This we write as

$$\nu(E) = \int_E f\, d\mu \quad \text{or} \quad d\nu = f\, d\mu$$

We now prove the main thm, which gives complete structure of signed measure relative to a +ve measure, Lemma: Let ν & μ be finite measures on (X,S). Then either $\nu \perp \mu$ or $\exists\ \epsilon > 0$ & $E \in S$ with $\mu(E) > 0$ (171) and $\nu \geq \epsilon\mu$ on E (i.e. E is a +ve set for $\nu - \epsilon\mu$).

Proof: Let $X = P_n \cup N_n$ be a Hahn decomposition of $\nu - \frac{1}{n}\mu$. Let $P = \cup P_n$, $N = \cap N_n = P^c$.

Then N is a -ve set for $\nu - n^{-1}\mu$ for all n. But then

$$0 \leq \nu(N) \leq \frac{1}{n}\mu(N) \to 0$$

That is, $\nu(N) = 0$.

If $\mu(P) = 0$, then $\nu \perp \mu$.

If $\mu(P) > 0$, then $\mu(P_n) > 0$ for some n & P_n is a +ve set for $\nu - n^{-1}\mu$.

Lebesgue - Radon - Nikodym thm

Let ν be a σ-finite signed measure and μ be a σ-finite positive measure on (X,S). Then $\exists!$ σ-finite signed measures λ, ρ on (X,S) s.t.

$\lambda \perp \mu$, $\rho \ll \mu$ & $\nu = \lambda + \rho$. (172)

Moreover, $\exists$ an integrable function $f: X \to \mathbb{R}$ s.t. $d\rho = f\,d\mu$.

Proof: Suppose that ν & μ be finite +ve measures. Let

$$\mathcal{F} = \left\{ f: X \to [0,\infty] : \int_E f\,d\mu \le \nu(E) \ \forall E \in S \right\}.$$

Then $\mathcal{F}$ is non-empty as $0 \in \mathcal{F}$.

If $f, g \in \mathcal{F}$, then $h = \max(f,g) \in \mathcal{F}$.

For this, let $A = \{x : f(x) > g(x)\}$.

For $E \in S$,

$$\int_E h\,d\mu = \int_{E\cap A} f\,d\mu + \int_{E\setminus A} g\,d\mu \le \nu(E\cap A) + \nu(E\setminus A) = \nu(E).$$

Let $a = \sup\left\{ \int_X f\,d\mu : f \in \mathcal{F} \right\}$. Then

$a \le \nu(X) < \infty$.

Choose seq^n $f_n \in \mathcal{F}$ s.t. $\int_X f_n\,d\mu \to a$.

Let $g_n = \max(f_1, \dots, f_n)$, & $f = \sup_n f_n$.

Then $g_n \in \mathcal{F}$, $g_n \uparrow f$ p.w.

and

$a \geq \int_X g_n \, d\mu \geq \int f_n \, d\mu, \quad \forall n \geq 1$

$\Rightarrow a = \lim \int_X g_n \, d\mu$. Hence by MCT, $a = \int_X f \, d\mu \Rightarrow f \in \mathcal{F}$.

$\left(\because \int_E f \, d\mu = \int_E \lim g_n \, d\mu = \lim \int_E g_n \, d\mu \leq \nu(E) \right)$

Since $f < \infty$ a.e., we can take f to be real-valued.

Then $d\lambda = d\nu - f \, d\mu$ is +ve measure since $f \in \mathcal{F}$. We claim $\lambda \perp \mu$.

If not, then by previous lemma, $\exists \ \epsilon > 0$, $E \in S$ with $\mu(E) > 0$ &

$$\lambda \geq \epsilon \mu \text{ on } E.$$

But $\epsilon \chi_E \, d\mu \leq d\lambda = d\nu - f \, d\mu$

That is $(f + \epsilon \chi_E) \, d\mu \leq d\nu \Rightarrow f + \epsilon \chi_E \in \mathcal{F}$

$\Rightarrow \int_X (f + \epsilon \chi_E) \, d\mu = a + \epsilon \mu(E) > a$,

which is a contradiction to the maximality of a. Thus $d\rho = f \, d\mu$.

For uniqueness, suppose

$$d\nu = d\lambda' + f' d\mu.$$

Then $d\lambda - d\lambda' = (f' - f)\,d\mu$.

But $\lambda - \lambda' \perp \mu$ and $(f' - f)\,d\mu \ll d\mu$

$\Rightarrow d\lambda - d\lambda' = (f' - f)\,d\mu = 0 \Rightarrow \lambda = \lambda'$ and $f' = f$ a.e. μ.

Case II If μ & ν are σ-finite, then X is countable disjoint union of μ-finite sets and a countable disjoint union of ν-finite sets. By intersection we obtain a disjoint seq^n $A_j \in S$ s.t. $\mu(A_j)$ and $\nu(A_j)$ are finite and $X = \dot{\cup} A_j$.

Define $\mu_j(E) = \mu(E \cap A_j)$,

& $\nu_j(E) = \nu(E \cap A_j)$.

Then by finite case, we have

$$d\nu_j = d\lambda_j + f_j\,d\mu_j, \text{ where } \lambda_j \perp \mu_j$$

Since $\mu_j(A_j^c) = \nu_j(A_j^c) = 0$, we have

$$\lambda_j(A_j^c) = \nu_j(A_j^c) - \int_{A_j^c} f_j\,d\mu_j = 0.$$

Hence, we can assume $f_j = 0$ on A_j^c.

Let $\lambda = \sum \lambda_j$; $f = \sum f_j$. Then (175)

$d\nu = d\lambda + f\,d\mu$, $\lambda \perp \mu$ and $d\lambda$ & $f\,d\mu$ are σ-finite.

The general case, when ν is signed measure can be obtained by considering ν^+ & ν^- etc.

$\nu = \lambda + \rho$, where $\lambda \perp \mu$, $\rho \ll \mu$ is known as Lebesgue decomposition of ν w.r.t. μ. In case, when $\nu \ll \mu$, then $d\nu = f\,d\mu$ for some f is known as Radon-Nikodym thm, and f is called Radon-Nikodym derivative of ν w.r.t μ, and we denote it by $\frac{d\nu}{d\mu}$.

i.e. $d\nu = \frac{d\nu}{d\mu} d\mu$.

<u>Chain rule:</u> Let ν be a σ-finite signed measure and μ & λ are σ-finite measures on (X, S) s.t. $\nu \ll \mu$ and $\mu \ll \lambda$.

(i) If $g \in L^1(\nu)$, then $g\left(\frac{d\nu}{d\mu}\right) \in L^1(\mu)$, and $\int g\, d\nu = \int g \frac{d\nu}{d\mu} d\mu$. (176)

(ii) We have $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \cdot \lambda \text{ a.e. } (\text{w.r.t. } \mu \text{ a.e.}).$$

Proof: By considering ν^+ & ν^- separately, we may assume $\nu \geq 0$. Then

(*) $\int \chi_E\, d\nu = \int \chi_E \left(\frac{d\nu}{d\mu}\right) d\mu, \ \forall E \in S.$

Hence (*) is true for any non-negative simple function in place of χ_E.

By MCT, (*) is true for any function in $L^1(\nu)$.

$$\Rightarrow \int_X g\, d\nu = \int_X g\left(\frac{d\nu}{d\mu}\right) d\mu$$

By replacing ν, μ by μ, λ and setting $g = \chi_E \frac{d\nu}{d\mu}$, we get

$$\int_E \frac{d\nu}{d\mu} d\mu = \int_E \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} d\lambda$$

$$\Rightarrow \frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}$$

Cor: If $\mu \ll \lambda$ & $\lambda \ll \mu$, then

$$\left(\frac{d\lambda}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right) = 1 \text{ a.e.}$$

Theorem: Let $1 < p < \infty$ & $\frac{1}{p} + \frac{1}{q} = 1$. Then for each $T \in (L^p)^*$, $\exists!\ g \in L^q$ s.t. $T(f) = \int fg\ \forall f \in L^p$.

If X is σ-finite, then the same conclusion is true for $p = 1$.

Proof: Suppose X is μ-finite. For $E \in S$, let $\nu(E) = T(\chi_E)$. Then ν is a signed measure on (X, S). Let $E = \bigcup_{j=1}^{\infty} E_j$. Then

$$\|\chi_E - \sum_{j=1}^{n} \chi_{E_j}\|_p = \left(\mu\left(\bigcup_{j=n+1}^{\infty} E_j\right)\right)^{1/p} \to 0.$$

$$\left(\because \mu(E) = \sum_{j=1}^{\infty} \mu(E_j) < \infty\right)$$

Since T is linear & cont.

$$\nu(E) = \sum T(\chi_{E_i}) = \sum \nu(E_i).$$

$\Rightarrow \nu$ is a signed measure.

If $\mu(E) = 0$, then $\chi_E = 0$ a.e. L^p.

$\Rightarrow \nu(E) = T(\chi_E) = 0.$

Hence $\nu \ll \mu$. By Radon Nikodym theorem, $\exists ! g \in L^1(\mu)$ s.t. $d\nu = g\,d\mu$.

Hence $T(\chi_E) = \nu(E) = \int_E g\,d\mu = \int \chi_E g\,d\mu$

$\Rightarrow T(\varphi_n) = \int \varphi_n g\,d\mu$

for every simple fn φ_n. So by lemma on page 156, $g \in L^q$ &

$\int |\varphi_n g| \leq \|T\| \cdot \|\varphi_n\|_p$.

Let $f \in L^p$. Then $\exists\ \varphi_n \xrightarrow{p.w} f$ & $|\varphi_n| \uparrow |f|$.

Hence $\int |\varphi_n g| < \int |fg| < \infty$ ($\because fg \in L^1$)

By DCT, $T(f) = \int fg\,d\mu$.

Suppose μ is σ-finite, and let $E_n \uparrow$ seq^n with $0 < \mu(E_n) < \infty$.

Notice that $L^p(E_n) \subset L^p(X)$ &

$L^q(E_n) \subset L^q(X)$.

Then by finite case $\exists !\ g_n \in L^q(E_n)$

s.t. $T(f) = \int f g_n$ & $\|g_n\|_q = \|T_{L^p(E_n)}\| \leq \|T\|$.

Since g_n is unique, $g_n = g_m$ on E_n if $n < m$. Thus, we can define g on X by setting $g = g_n$ on E_n. By MCT

$$\|g\|_q = \lim \|g_n\|_q \le \|T\| < \infty$$

$\Rightarrow g \in L^q(X)$.

Now, if $f \in L^p(X)$, then, by DCT,

$$f\chi_{E_n} \to f \text{ in } L^p(X).$$

Hence,

$$T(f) = \lim T(f\chi_{E_n}) = \lim \int f\chi_{E_n} g = \int fg$$

(by DCT).

Finally, suppose μ is arbitrary, & $p > 1$. So $q < \infty$. Then for each σ-finite set $E \subset X$, $\exists!$ $g_E \in L^q(E)$ s.t. $T(f) = \int f g_E$, $\forall f \in L^p(E)$ & $\|g_E\| \le \|T\|$.

If F is a σ-finite set & $F \supset E$, then $g_F = g_E$ on E. So

$$\|g_F\|_q \geq \|g_E\|_q.$$

Let $M = \sup\{\|g_E\|_q : E \text{ is } \sigma\text{-finite}\}$.

Then $M \le \|T\|$. Choose a seq^n E_n s.t.

$\|g_{E_n}\|_q \to M$. Set $F = \bigcup_{n=1}^{\infty} E_n$, then F is σ-finite, and

$$\|g_F\|_q \geq \|g_{E_n}\|_q \quad \forall\, n \geq 1.$$

$$\Rightarrow \|g_F\|_q = M.$$

Now, if A is a σ-finite set, ~~then~~ and $A \supseteq F$, then

$$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \leq M^q = \int |g_F|^q.$$

$$\Rightarrow g_{A\setminus F} = 0 \text{ a.e.} \Rightarrow g_A = g_F \text{ a.e.}$$

But if $f \in L^p(X)$, then

$$A = F \cup \{x : f(x) \neq 0\} \text{ is } \sigma\text{-finite.}$$

So $T(f) = \int f g_A = \int f g_F = \int f g$,

$(\because f_{A^c} = 0)$

where $g = g_F$.

Note that the case $p=1$, with μ σ-finite, is similar to $p>1$ case.