## Product measure:

Let $(X, S, \mu)$ and $(Y, T, \nu)$ be two measure spaces. For $A \in S$, and $B \in T$, we say $A \times B$ is a measurable rectangle.

Let $Q$ be the collection of all finite union of rectangles from $S \times T$. Then $Q$ is an algebra of sets from $S \times T$. Now, for

$$A \times B = \bigcup_{i,j=1}^{\infty} A_i \times B_j,$$ we can write

$$\chi_A(x)\chi_B(y) = \chi_{A\times B}(x,y) = \sum_{i,j=1}^{\infty} \chi_{A_i \times B_j}(x,y).$$

Integrating both the sides, first w.r.t. $x$ and then w.r.t. $y$ and using Beppo-Levi theorem, we get

$$\mu(A)\nu(B) = \sum_{i,j=1}^{\infty} \mu(A_i)\nu(B_j). \quad \text{---} (*)$$

Now, for $E \in Q$ with $E = \bigcup_{i,j=1}^{n,m} A_i \times B_j$, define

$$\eta_0(E) := \sum_{j=1}^{m}\sum_{i=1}^{n} \mu(A_i)\nu(B_j).$$

Then by $(*)$, $\eta_0$ is a pre-measure on $Q$.

Hence, $\eta_0$ generates an outer measure $\eta^*$ on $P(X \times Y)$. Let

$$S \otimes T = \{\eta^*\text{-measurable sets in } X \times Y\}$$
$$= \text{the smallest } \sigma\text{-algebra containing } Q.$$

Then $\eta = \eta^*/_{S\otimes T} : S\otimes T \longrightarrow [0,\infty]$ is a measure on $S\otimes T$.



Notice that if $(X, S, \mu)$ & $(Y, T, \nu)$ are $\sigma$-finite, then $X = \cup E_n$, $Y = \cup F_m$, where $\mu(E_n) < \infty$ & $\nu(F_m) < \infty$. Hence

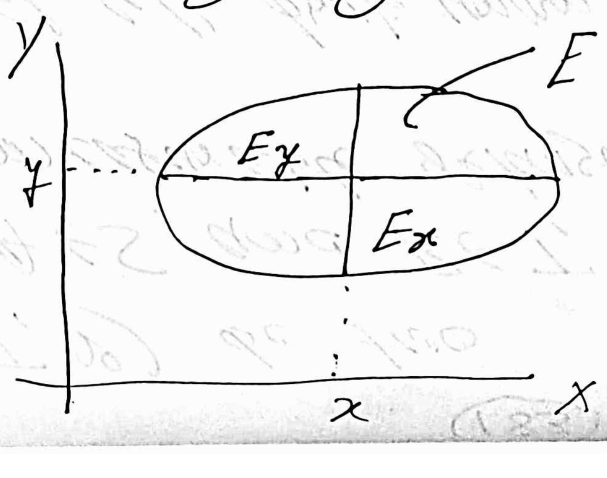
$$\eta(X\times Y) = \sum\sum \eta(E_n \times F_m) = \sum\sum \eta_0(E_n\times F_m)$$
$$= \sum\sum \mu(E_n)\,\nu(F_m).$$

Hence, $\eta$ will be a unique extension of $\eta_0$ such that $\eta(A\times B) = \mu(A)\,\nu(B)$, whenever $A\times B \in R$. (See the previous result page 73.)

In this case, we write $\eta = \mu\times\nu$. Hence $(X\times Y, S\otimes T, \mu\times\nu)$ is a measure space.

Note that, if $E \in S\otimes T$, then $E$ need not be a rectangle, but we need to calculate $(\mu\times\nu)(E)$!

For this, we define projection (or Section) of $E$ on $X$ & $Y$, in the following way.




For $(x,y) \in X\times Y$, let

$E_x = \{y \in Y : (x,y) \in E\}$

and $E^y = \{x \in X : (x,y) \in E\}$.

Monotone class: A collection of sets in $X$ which is closed under countable increasing union and countable decreasing intersection is called monotone class. That is $\mathcal{M}$ is a monotone class if $A_i \uparrow$ & $B_j \downarrow$ seq$^n$s in $\mathcal{M}$, implies $\bigcup A_i$ & $\bigcap B_j \in \mathcal{M}$. (185)

Ex. Every $\sigma$-algebra is a monotone class, but converse need not be true. However, if $\mathcal{A}$ is an algebra of sets in $X$. Then the monotone class generated by $\mathcal{A}$ coincides with the $\sigma$-algebra generated by $\mathcal{A}$.

Let $S(\mathcal{A})$ = $\sigma$-algebra generated by $\mathcal{A}$ and $M(\mathcal{A})$ = monotone class generated by $\mathcal{A}$.

Theorem (Monotone class theorem):

$$S(\mathcal{A}) = M(\mathcal{A}).$$

Proof: Since $S(\mathcal{A})$ is a monotone class containing $\mathcal{A}$, $M(\mathcal{A}) \subset S(\mathcal{A})$. On the other hand we need to show that $M(\mathcal{A})$ is a $\sigma$-algebra, then $M(\mathcal{A}) \supset S(\mathcal{A})$.

For this, let $E \in M(\mathcal{A}) = M$ (say) and define

$M(E) = \{F \in M : E \setminus F,\ F \setminus E,\ E \cap F \text{ are in } M\}$

(i) $\phi, E \in M(E)$ ($\because E \in M$).

(ii) $M(E)$ is a monotone class,

if $F_n \in M(E)$ & $\uparrow$, then $F_n \setminus E \in M$ & $\uparrow$

$\Rightarrow \bigcup_{n=1}^{\infty} F_n \setminus E = \bigcup_{n=1}^{\infty} (F_n \setminus E) \in M(E)$ etc.

(iii) If $E \in \mathcal{A}$, then $F \in M(E)$, $\forall F \in \mathcal{A}$,

because $\mathcal{A}$ is an algebra.

i.e. $E \in \mathcal{A} \Rightarrow \mathcal{A} \subset M(E)$. Again, since $M(E)$ is a monotone class,

$$M = M(\mathcal{A}) \subset M(E), \ \forall E \in \mathcal{A}.$$

Notice that $E \in M(F)$ iff $F \in M(E)$ (by defn).

Hence for $F \in M$, $F \in M(E)$, $\forall E \in \mathcal{A}$

$$\Rightarrow E \in M(F), \ \forall E \in \mathcal{A}$$

$$\Rightarrow \mathcal{A} \subset M(F), \ \forall F \in M$$

$$\Rightarrow M \subset M(F), \ \forall F \in M.$$

Thus, if $E, F \in M$, then $E \setminus F$, $E \cap F \in M$.

Since $\mathcal{A} \subset M$ & $X \in \mathcal{A}$, it follows that $M$ is closed under union and complement. That is, $M$ is an algebra.

Now, if $\{A_i\}_{i=1}^{\infty} \subset M$, then $\bigcup_{i=1}^{n} A_i \in M$ & $\uparrow$,

but $M$ is a monotone class, implies

$\bigcup_{i=1}^{\infty} A_i \in M$. That is, $M$ is a $\sigma$-algebra

and $M(\mathcal{A}) = S(\mathcal{A})$.

**Theorem**: Let $(X, S, \mu)$ and $(Y, T, \nu)$ be two $\sigma$-finite measure spaces. Suppose $E \in S \otimes T$, then (i) $E_x \in S$, and $E^y \in T$, $\forall (x, y) \in X \times Y$.

(ii) $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable functions on $(X, S)$ and $(Y, T)$ respectively.

(iii) $(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y)$.

Proof: The proof of this result is based on monotone class theorem.

(i) Let $\mathcal{B} = \{E \in S \otimes T : E_x \in S, E^y \in T, \forall (x, y) \in X \times Y\}$.

Then $\mathcal{B} \subseteq S \otimes T$. On the other hand, it is easy to see that $\mathcal{B}$ is a $\sigma$-algebra containing $\mathcal{R}$. Hence $S \otimes T \subseteq \mathcal{B}$. This proves (i).

Proof of (ii) & (iii): Let $\mu$ & $\nu$ be both finite measure.

Let $\mathcal{P} = \{E \in S \otimes T : \text{(ii) \& (iii) hold}\}$.

For $E = A \times B$, $A \in S$, $B \in T$

$\nu(E_x) = \chi_A(x)\, \nu(B)$ and $\mu(E^y) = \mu(A)\, \chi_B$.

and $(\mu \times \nu)(E) = \mu(A)\nu(B) = \int \nu(E_x)\, d\mu(x)$

$$= \int \mu(E^y)\, d\nu(y).$$

Thus, theorem is true for $E = A \times B$. Hence, it will hold for finite disjoint union of rectangles. Hence by the monotone class theorem, it is suffice to show that $\mathcal{P}$ is a monotone class.

If $E_n$ is an $\uparrow$ seq$^n$ in $\mathcal{P}$, and $E = \bigcup_{n=1}^{\infty} E_n$, then $\mu((E_n)^y)$ is measurable function that increases p.w. to $\mu(E^y)$. Hence $\mu(E^y)$ is measurable and by MCT, (188)

$$(*) \quad \int \mu(E^y)\, d\nu(y) = \lim \int \mu((E_n)^y)\, d\nu(y)$$
$$= \lim (\mu \times \nu)(E_n) = (\mu \times \nu)(E).$$

Likewise, $\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x)$. So $E \in \mathcal{P}$.

Similarly, if $E_n \downarrow$ seq$^n$ in $\mathcal{P}$ and $E = \bigcap_{n=1}^{\infty} E_n$, then $y \mapsto \mu((E_1)^y) \in L^1(\nu)$, because, $\mu((E_1)^y) \leq \mu(X) < \infty$ and $\nu(Y) < \infty$.

Since $\mu((E_n)^y) \downarrow \mu((E_1)^y)$. By DCT, we can see that $E \in \mathcal{P}$.

Finally, if $\mu$ and $\nu$ are $\sigma$-finite, then $X \times Y$ can be written as the union of an increasing seq$^n$ $\{X_i \times Y_j\}$ of rectangles of finite measure. By previous argument applied on $E \cap (X_i \times Y_i)$, we get

$$\mu \times \nu(E \cap (X_i \times Y_i)) = \int \chi_{X_i}(x)\, \nu(E_x \cap Y_i)\, d\mu(x)$$

$$= \int \chi_{Y_i}(y)\, \mu(E^y \cap X_i)\, d\nu(y).$$ 189

By MCT, we get the required result.

Ex. Let $f : X \xrightarrow{\text{m'ble}} \mathbb{R}$. Then we can define

$$\varphi : X \times \mathbb{R} \longrightarrow \mathbb{R}^2 \xrightarrow{\ \psi\ } \mathbb{R} \text{ by}$$

$$\varphi(x,y) = (f(x), y) \text{ and } \psi(\xi, \eta) = \xi - \eta.$$

$$\varphi^{-1}\{(a,b) \times (c,d)\} = \{(x,y) \in X \times \mathbb{R} : \varphi(x,y) \in (a,b) \times (c,d)\}$$

$$= \{(x,y) : a < f(x) < b,\ c < y < d\}$$

$$= f^{-1}\{(a,b)\} \times (c,d) \text{ is a m'ble subset of } X \times \mathbb{R}.$$

Hence $\psi \circ \varphi$ is m'ble. Consider

$$(\psi \circ \varphi)^{-1}(0) = \{(x,y) \in X \times \mathbb{R} : (\psi \circ \varphi)(x,y) = 0\}$$

$$= \{(x,y) : \psi(f(x), y) = 0\}$$

$$= \{(x,y) : y = f(x),\ x \in X\}$$

$$= G_f, \text{ the graph of } f.$$

Hence, the graph of a measurable function is m'ble.

Theorem (Tonelli): Let $(X, S, \mu)$ and $(Y, T, \nu)$ are σ-finit.

Let $f: X \times Y \to [0, \infty]$ be a $S \otimes T$-m'ble function. Then for fixed $(x_0, y_0) \in X \times Y$,

(i) $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are measurable functions on $(X, S)$ and $(Y, T)$ respectively.

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(ii) $y \mapsto \int_X f(x,y)\, d\mu(x)$ and $x \mapsto \int_Y f(x,y)\, d\nu(y)$ are m'ble.

(iii) $$\int_{X \times Y} f(x,y)\, d(\mu \times \nu)(x,y) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y)$$
$$= \int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x).$$

Proof: Since $f$ is $S \otimes T$-m'ble on $X \times Y$, $\exists$ a seq$^n$ $\varphi_n$ of simple function that increases to $f$ point-wise. Hence,

$\lim \varphi_n(x_0, y) = f(x_0, y) \longleftarrow y$ } are m'ble.
& $\lim \varphi_n(x, y_0) = f(x, y_0) \longleftarrow x$ }

Now, by MCT,

(1) -- $y \mapsto \int_X f(x,y)\, d\mu(x) = \lim \int_X \varphi_n(x, y)\, d\mu(x)$

$$= \lim_{n \to \infty} \sum_{j=1}^{k_n} \alpha_j\, \nu\{(E_j)_y\},$$

where $\varphi_n = \sum_{j=1}^{k_n} \alpha_j \chi_{E_j}$. This proves (ii).

Further $(\varphi_n)_y = \sum \alpha_j\, \nu((E_j)_y)$ ↑ sequence, By applying MCT in (1), we get

$$\int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y) = \lim \int_Y \int_X \varphi_n(x,y)\, d\mu(x)\, d\nu(y).$$

$$= \lim \int_{X\times Y} \varphi_n(x,y)\, d(\mu\times\nu)(x,y).$$

Similarly, other equality follows.

<u>Fubini's theorem</u>:

Let $(X,S,\mu)$ and $(Y,S,\nu)$ be two $\sigma$-finite measure spaces. If $f \in L^1(\mu\times\nu)$, then (i) $x \mapsto f(x,y)$ & $y \mapsto f(x,y)$ are a.e. integrable on $X$ & $Y$ respectively.

(ii) $y \mapsto \int_X f(x,y)\, d\mu(x)$ and $x \mapsto \int_Y f(x,y)\, d\nu(y)$ are integrable on $Y$ and $X$, respectively.

(iii) $$\int_{X\times Y} f(x,y)\, d(\mu\times\nu)(x,y) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y)$$

$$= \int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x).$$

Proof: Let $f = f^+ - f^-$, then $f^+, f^- \in L^1(\mu\times\nu)$ and are non-negative. Hence, by linearity of integral on $L^1$,

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_{X\times Y} f^+\, d(\mu\times\nu) - \int_{X\times Y} f^-\, d(\mu\times\nu).$$

Hence by Tonelli's theorem,

$$(*) \quad \int_X \int_Y f^+(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X f^+\, d\mu(x)\, d\nu(y) < \infty.$$

$\Rightarrow \int_Y f^+(x,y)\, d\nu(y)$ & $\int_X f^+(x,y)\, d\mu(x)$ are finite a.e. w.r.t $\mu$ and $\nu$ resp. and integrable w.r.t $\mu$ and $\nu$ resp. This proves (i) and (ii). Hence by Tonelli theorem and $(*)$, we get (iii).

Remark: If the measure space $(X, S, \mu)$ and $(Y, T, \nu)$ are complete, their product $(X \times Y, S \otimes T, \mu \times \nu)$ need not be complete.
Suppose $A \subset X$, $A \neq \phi$, $\mu(A) = 0$. let $B \subset Y$ but $B \notin T$, then $A \times B \subset A \times Y$, but
$$\eta^*(A \times B) \leq \eta^*(A \times Y) = \mu(A)\nu(Y) = 0,$$
however, $A \times B \notin S \otimes T$.

Ex. Let $m_1^*$ and $m_2^*$ denote the usual Lebesgue outer measure on $\mathbb{R}$ and $\mathbb{R}^2$ resp. and $m_1, m_2$ are their corresponding Lebesgue measures. Then $m_1 \times m_1$ is not a complete measure, however, $m_2$ is complete. Though, completion of $m_1 \times m_1$ is $m_2$.
let $R_2 = \{ [a,b) \times [c,d) : a, b, c, d \in \mathbb{R} \}$, and

$\mathcal{B}_2$ is the $\sigma$-algebra generated by $\mathcal{R}_2$. Then $\mathcal{B}_2$ is nothing but Borel $\sigma$-algebra on $\mathbb{R}^2$.

Since $\mathcal{R}_2 \subset \mathcal{M}_1 \otimes \mathcal{M}_1$, it follows that

$$\mathcal{B}_2 \subset \mathcal{M}_1 \otimes \mathcal{M}_2 \quad (\because \mathcal{M}_1 \otimes \mathcal{M}_2 \text{ is a } \sigma\text{-alg.})$$

Further, $\mathcal{R}_2 \subset \mathcal{M}_2$ and $\mathcal{M}_1 \otimes \mathcal{M}_1$ is the smallest $\sigma$-algebra containing $\mathcal{R}_2$, hence (193)

$$\mathcal{B}_2 \subset \mathcal{M}_1 \otimes \mathcal{M}_1 \subset \mathcal{M}_2.$$

But completion of $\mathcal{B}_2$ is $\mathcal{M}_2$. So, if $E \in \mathcal{M}_2$, then $\exists$ $F, G \in \mathcal{B}_2$ with $F \subset E \subset G$ such that $m_2(G \setminus F) = 0$. Thus,

$$m_1 \times m_1 (E \setminus F) \leq m_1 \times m_1 (G \setminus F) = m_2(G \setminus F) = 0.$$

$$(m_1 \times m_1)(E) = (m_1 \times m_1)(F) = m_2(F) = m_2(G).$$

Since $m_2(F) \leq m_2(E) \leq m_2(G)$.

$$\Rightarrow (m_1 \times m_1)(E) = m_2(E).$$

Ex. Let $\mathcal{B}(\mathbb{R}^2)$ be the $\sigma$-algebra generated by open sets (or open rectangles in $\mathbb{R}^2$). Then $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Since $(a,b) \times (c,d) \subset \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$, It follows that

$$\mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$$

On the other hand, $(a,b) \times (c,d) \subset \mathcal{B}(\mathbb{R}^2)$

By varying $(a,b)$ & fix $(c,d)$, we get

$$\mathcal{B}(\mathbb{R}) \times (c,d) \subset \mathcal{B}(\mathbb{R}^2)$$

$$\Rightarrow \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^2).$$

But then, $\sigma(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})) \subseteq \mathcal{B}(\mathbb{R}^2)$

that is, $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^2)$

($\because$ $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ is ~~the~~ smallest $\sigma$-algebra containing $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$).

Thus, $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.