Some notation and def^n on $\mathbb{R}^n$

1

For $n \in \{1, 2, \cdots\} = \mathbb{N}$,

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{-copies}}$$

$x \in \mathbb{R}^n$, $x = (x_1, x_2, \cdots, x_n)$

$0 \in \mathbb{R}^n$, $0 = (0, 0, \cdots, 0)$.

$x, y \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$

$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$.

$\alpha x = (\alpha x_1, \alpha x_2, \alpha x_n)$.

$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Then $\langle \cdot, \cdot \rangle$ satisfies,

(i) $\langle x, x \rangle = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$

(ii) $\langle x, x \rangle = 0$ iff $x = 0$.

(iii) for $\alpha, \beta \in \mathbb{R}$ & $x, y, z \in \mathbb{R}^n$

$$\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$$

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$$

($\langle \cdot, \cdot \rangle$ is a bilinear map and is called inner product)

$\textcircled{2}$

~~Write For $x \in \mathbb{R}^n$,~~ $\|x\| = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + \cdots + x_n^2}$

For $x \in \mathbb{R}^n$, write $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \cancel{= \sqrt{\langle x, x\rangle}}$.

Then $\|x\| = \sqrt{\langle x, x\rangle}^{1/2}$.

If $x, y \in \mathbb{R}^n$, then

$$|\langle x, y\rangle| \leq \|x\| \|y\|.$$

(Cauchy Schwarz inequality)

if $x \neq 0$, $y \neq 0$, $\|x\| \neq 0$, $\|y\| \neq 0$.

$$\left|\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle\right| \leq 1$$

But $\left\|\frac{x}{\|x\|}\right\| = 1$ & $\left\|\frac{y}{\|y\|}\right\| = 1$.

we need to prove only when $\|x\| = 1 = \|y\|$.

~~For $t \in \mathbb{R}$,~~ $\langle x - ty, x - ty\rangle = \|x - ty\|^2 \geq 0$.

$\forall t \in \mathbb{R}$

Let $p(t) = \langle x - ty, x - ty\rangle$. Then

$$p(t) = \langle x, x\rangle - 2t\langle x, y\rangle + t^2\langle y, y\rangle$$

$$= t^2 - 2t\langle x, y\rangle + 1 \geq 0$$

Take $t_0 = \langle x, y\rangle$, $p(t_0) \geq 0$

$\cancel{\langle x, y\rangle}$ $t_0^2 - 2t_0^2 + 1 > 0$

$t_0^2 \leq 1 \Rightarrow |t_0| \leq 1$ ③

ie $|\langle x, y\rangle| \leq 1$.

Notice $|\langle x,y\rangle| = 1$ iff ~~$y = \alpha y$~~ ~~$x = \alpha y$~~ ~~$y = \alpha x$~~ for some $\alpha \in \mathbb{R}$.

Suppose $y = \alpha x$. then $x = \alpha y$ $\Rightarrow |\alpha| = 1$

$|\langle x, \alpha x\rangle| = |\alpha| |\langle x, x\rangle| = |\alpha| \cdot 1 \cdot 1$

$= \|\alpha x\| \cdot \|x\| = \|y\| \cdot \|x\|$.

$\Rightarrow |\langle x, \alpha x\rangle| = 1.$

Suppose $\underbrace{|\langle x,y\rangle| = 1}_{(1)}$. Claim $y = \alpha x$ for some α.

$p(t) = t^2 - 2t\langle x, y\rangle + 1$. If we take

$t_0 = \langle x, y\rangle$, then ~~$p(t_0)$~~

$p(t_0) = \langle x, y\rangle^2 - 2\langle x, y\rangle^2 + 1$

$= 0$ (by (1))

But $p(t_0) = \|x - t_0 y\|^2 = 0$ iff

$x = t_0 y$.

Result: $|\langle x, y\rangle| \leq \|x\| \|y\|$, $\forall x, y \in \mathbb{R}^n$.

and $|\langle x, y\rangle| = \|x\| \|y\|$ iff $\exists\ \alpha \in \mathbb{R}$ s.t.

$x = \alpha y$. (linearly dependent)

(Explain linearly dependent sets etc)

For $x, y \in \mathbb{R}^n$,

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$
$$= \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle$$
$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$$
$$\left(\because |\langle x, y\rangle| \leq \|x\|\|y\|\right) \quad \text{C.S.}$$
$$= \left(\|x\|^2 + \|y\|\right)^2 \qquad \left(\because \begin{array}{l} a, b \geq 0 \\ a^2 \leq b^2 \Rightarrow a \leq b \end{array}\right)$$
$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$
(Triangle inequality)

Infimum & Supremum :

$\mathbb{R} = (-\infty, \infty)$, $A \subseteq \mathbb{R}$

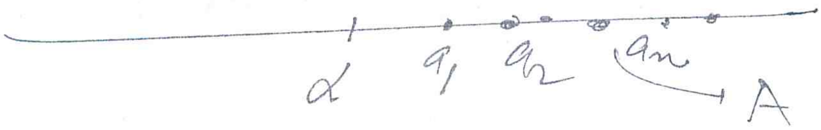
$a, b \in \mathbb{R}$, either $a \leq b$ or $b \leq a$.

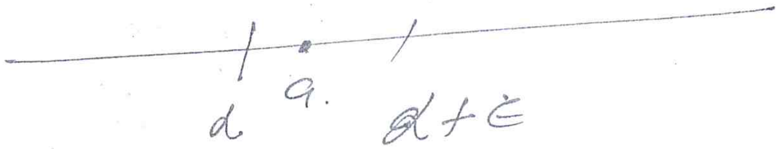
Then one can think of maxi & min of the set A. $A = \{1, 2, \ldots, 100\}$, $\max A = 100$, $\min A = 1$. $A = \mathbb{N} = \{1, 2, \ldots\}$

$\min A = 1$ but $\max A = ?$

$A = \{\ldots, -2, -1, 0\}$

$\max A = 0$, $\min A = ?$





"Infimum (or greatest lower bound)

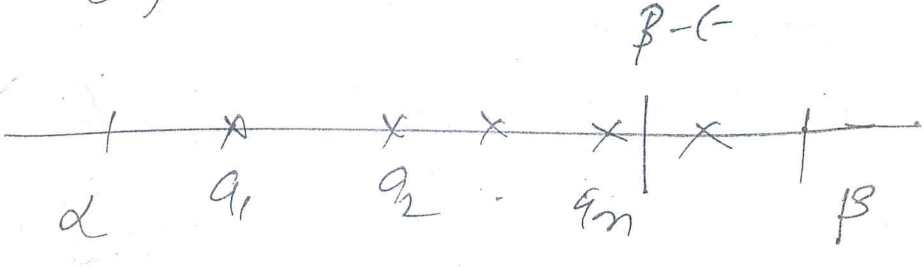
Def^n: $\alpha \in \mathbb{R}$ is called infimum (or glb) of set $A \subset \mathbb{R}$, if (i) $\alpha \leq a$, $\forall\, a \in A$

(ii) $\forall\, \epsilon > 0$, $\exists\, a_\epsilon \in A$ s.t.

$$\alpha + \epsilon > a_\epsilon$$

($\alpha+\epsilon$ is not a lower bound)

*(ii) $\beta \in \mathbb{R}$ is supremum. . . . .



if (i) $\beta \geq a$, $\forall\, a \in A$

(ii) $\forall\, \epsilon > 0$, $\beta - \epsilon$ is not an upper bound of A. That is, $\exists\, b_\epsilon \in A$

s.t. $\beta - \epsilon < b_\epsilon$

Notations:

$\inf A = \alpha$, $\sup A = \beta$.

$\Rightarrow A \subset [\alpha, \beta]$

Bolzano - Weierstrass theorem:

Every bdd sequence $(a_n) \subset \mathbb{R}$ has a conv. subsequence.

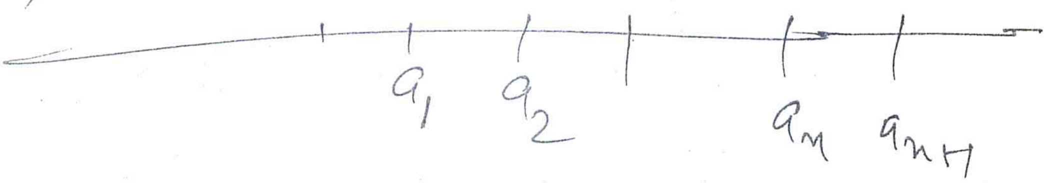
$f : \mathbb{N} \longrightarrow \mathbb{R}$

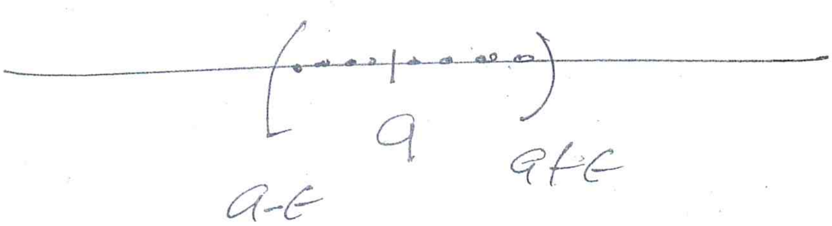
$\{f(1), f(2), \ldots \}$

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

Conv. of seqn:

$a_n \longrightarrow a$





$a_n \in (a-\epsilon, a+\epsilon)$, $n \geq N$

$a-\epsilon < a_n < a+\epsilon \Rightarrow |a_n - a| < \epsilon, \ \forall n \geq N.$

Ex. $a_n = \frac{1}{n}$. expect what would be the limit?

Necessary facts about conv. seq^n:

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st

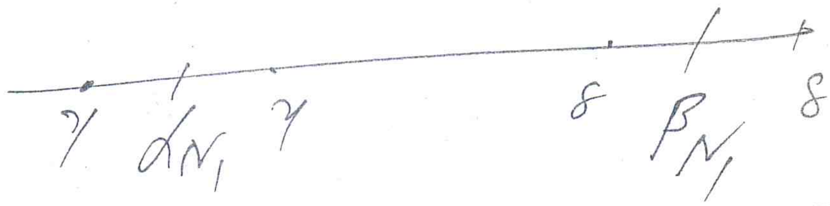
$$|a_n - a| < \epsilon, \quad \forall n \geq N$$

$$a_n \in (a-\epsilon, a+\epsilon), \quad \forall n \geq N$$

put $\epsilon = 1$, $a_n \in (a-1, a+1)$ $\forall n \geq N_1$

$$\left.\alpha_{N_1} = \inf_{n \geq N_1} a_n \leq a-1, \quad \beta_{N_1} = \sup_{n \geq N_1} a_n \leq a+1. \right\} (**)$$

$$\gamma = \inf\{a_1, \dots, a_{N_1}\}, \quad \delta = \sup\{a_1, \dots, a_{N_1}\}$$



$$\alpha = \min\{\gamma, \alpha_{N_1}\}, \quad \beta = \max\{\delta, \beta_{N_1}\}$$

$\Rightarrow a_n \in [\alpha, \beta]$.

$\Rightarrow \{a_n\}_{n=1}^{\infty}$ is a bdd set (or seq^n)

(i) If a_n is conv, then a_n is bdd.

In (*), $\epsilon = 1, 2, 3, \dots$ or $\epsilon \in (0, \infty)$

$\{\alpha_{N_\epsilon} : \epsilon > 0\}$ — a family of lower bo

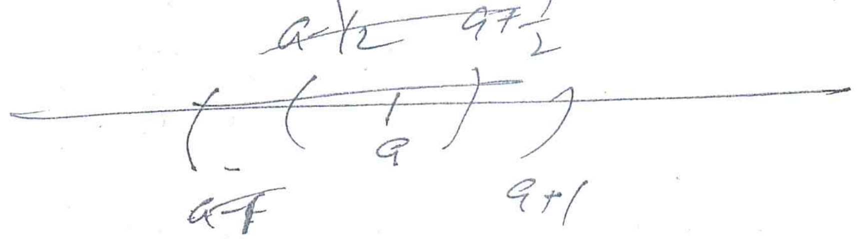
$\{\beta_{N_\epsilon} : \epsilon > 0\}$ → a family of upper bounds.

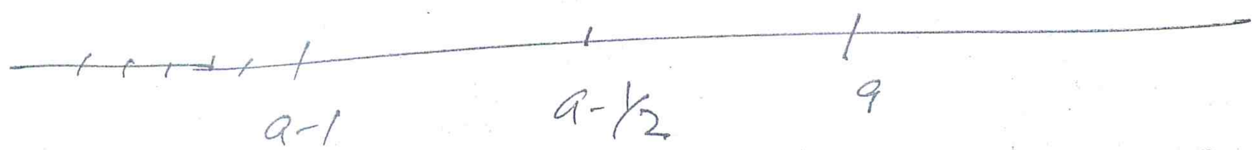
$\alpha_{N_1} \leq \inf_{n \geq N_1} a_n < a - \epsilon$ (ϵ very small)

$\epsilon = 1/2$, ~~$\alpha_{N_1} < \alpha_{N_{1/2}} < a - 1/2$~~

~~$\alpha_{N_{1/2}} < a - \frac{1}{2}$~~

$\alpha_{N_{1/2}} < \alpha_{N_1}$





$\alpha_{N_1} \leq \alpha_{N_{1/2}} < a - \frac{1}{2}$

we $\alpha_{N_1} \leq \alpha_{N_{1/2}} < \cdots < \alpha_{N_{1/2^k}} < \cdots < a - \frac{1}{2^k}$ (s↑

$\boxed{\alpha_{N_k} < a - \frac{1}{2^{k-1}}}$

$\sup_{k} \alpha_{N_1} \geq a$ (?)

$\sup_{k \geq 1} \left(\inf_{n \geq N_k} a_n \right) \geq a.$

$\underline{\lim}\, a_n \leq a \leq \underline{\lim}\, a_n$

Now,
$$\sup_{n\geq k} a_n - \inf_{n\geq k} a_n \geq 0$$

$$\inf_{k\geq 1}\left(\sup_{n\geq k} a_n - \inf_{n\geq k} a_n\right) \geq 0$$

$$\inf_{k\geq 1}\left(\sup_{n\geq k} a_n\right) - \sup_{k\geq 1}\left(\inf_{n\geq k} a_n\right) \geq 0$$

$$\overline{\lim}\, a_n \geq \underline{\lim}\, a_n.$$

Note: $\sup_{a\in A}\{-a\} = -\inf_{a\in A} a.$

$$\sup(-A) = -\inf A.$$

(i) $\sup A = \alpha$, $a \leq \alpha$, $\forall a\in A$

$$-a \geq -\alpha, \forall a\in A$$

$$\inf(-A) \geq -\alpha$$

$$-\inf(-A) \leq \alpha = \sup A. \quad \text{—(1)}$$

(ii) $\inf(-A) = \beta$, $-a \geq \beta$, $\forall a\in A$

$$a \leq -\beta, \quad \forall a\in A$$

$$\sup A \leq -\beta = -\inf(-A) \quad \text{—(2)}$$

$$-\inf(-A) \leq \sup A \leq -\inf(-A).$$

$$\inf(-A) = -\sup A.$$

Return to Inf:

$\inf A = \alpha$

(i) $a \geq \alpha, \ \forall a \in A$

(ii) $\forall \epsilon > 0, \ \exists \, a_\epsilon \in A$ s.t.

$\alpha + \epsilon > a_\epsilon$

Put $\epsilon = \frac{1}{n}$,

$\alpha + \frac{1}{n} > a_n > \alpha > \alpha - \frac{1}{n}$

$\Rightarrow |a_n - \alpha| < \frac{1}{n}$ ~~→0~~, $\forall n \geq 1$

$< \epsilon$, for $n \geq N$

$\Rightarrow |a_n - \alpha| < \epsilon, \ \forall n \geq N.$

$\alpha = \lim a_n$.

Bolzano-Weierstrass theorem:

Every bounded set ~~A~~ in $\mathbb{R}^n$ ~~(...)~~ has a conv. subseqn.

~~$\inf A$~~ $\inf\limits_{n \geq 1} a_n = \alpha, \quad \sup\limits_{n \geq 1} a_n = \beta$

$a_n \in [\alpha, \beta]$,

(i) $a_n \geq \alpha, \ \forall n \geq 1$

$$\inf\{a_1, a_2, \ldots, a_n, \ldots\} = \alpha$$

$$a_{n_1}, a_{n_2}, \ldots \ldots \rightarrow \alpha$$

$$\lim_{k \to \infty} a_{n_k} = \alpha.$$

B-W theorem for $\mathbb{R}^2$ ($\cancel{n \geq 2}$).

$$\{X_n\} = \{(x_n, y_n)\}$$

$$\|X_n\| = \sqrt{x_n^2 + y_n^2} \leq M, \quad \forall n \geq 1$$

$$|x_n| \leq \sqrt{x_n^2 + y_n^2} \leq M$$

$$|y_n| \leq \sqrt{x_n^2 + y_n^2} \leq M$$

$$x_{n_k} \rightarrow x \text{ (B-W)}, \quad \{(x_{n_k}, y_{n_k})\} \text{ is bdd.}$$

$$y_{n_k} \text{ is bdd} \Rightarrow y_{n_{k,l}} \rightarrow y \text{ (B-W)}$$

$$(x_{n_{k,l}}, y_{n_{k,l}}) \rightarrow (x, y)$$

Note: $|a_n - a| < \epsilon, \quad \forall n \geq N$

$a_N \quad a_{N+1} \quad a_{N+2}$

$a_{n_k} \quad |a_{n_k} - a| < \epsilon, \quad n_k > N$

$\mathbb{R}^n$: $X_k = (x_1^k, x_2^k, \ldots, x_n^k)$

if $\{X_k\}$ is bdd seqn in $\mathbb{R}^n$

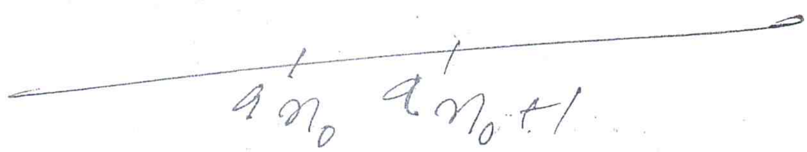
$\exists$ subseqn $X_{k_l} \to X \in \mathbb{R}^n$.

Ex. $a_n \in \mathbb{R}$, $\{a_n\} \uparrow$ & bdd above, then a_n is conv. & converges to $\sup a_n$.

let $\sup\limits_{n \geq 1} a_n = \beta$

For $\epsilon > 0$, $\exists$ a_{n_0} st

$$\beta - \epsilon < a_{n_0} < a_{n_0+1} \cdots$$



$$\beta - \epsilon < a_n, \ \forall n \geq n_0$$

$$< \beta < \beta + \epsilon.$$

$$\Rightarrow a_n \to \beta.$$

Ex if $a_n \downarrow$ & bounded below, then

$a_n \to \inf a_n$.

Closed sets & open sets in $\mathbb{R}$:

Def^n: A set $F \subset \mathbb{R}$ is said to be closed if ~~any~~ seq^n $a_n \in F$ s.t $a_n \to a \in \mathbb{R} \Rightarrow a \in F$.

(i.e limit of any seq^n in F is contained F).

A set $O \subset \mathbb{R}$ is said to be open in $\mathbb{R}$ if O^c is closed.

It follows that O is open if for any $x \in O$, $\exists\ \epsilon > 0$ s.t $(x-\epsilon, x+\epsilon) \subset O$.

If not for some $x \in O$, $\nexists\ \epsilon > 0$ s.t

$$(x-\epsilon, x+\epsilon) \not\subseteq O.$$

$\epsilon = 1/n$, $\exists\ x_n \in (x-\frac{1}{n}, x+\frac{1}{n}) \cap O^c$

$\Rightarrow x_n \to x$ and O^c is closed.

$\Rightarrow x \in O^c$ ~~absured~~ absured.

Note: F is closed iff F^c is open.

Def^n: A set $O \subset \mathbb{R}$ is open if for any $x \in O$, $\exists\ \epsilon > 0$ st $(x-\epsilon, x+\epsilon) \subset O$.

$\to$ **

Ex $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, A is not open & not closed.

All $I = (a, b)$ are open

(i) $\bigcup_{i \in I} O_i$ is open, O_i - open

(ii) $\bigcap_{i \in I} F_i$ is closed, F_i - closed.

(iii) $\bigcap_{i=1}^{n} O_i$ is open. O_i - open

(iv) $\bigcup_{i=1}^{n} F_i$ is closed, F_i - closed.

$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ Ex. $(-\frac{1}{n} < x < \frac{1}{n}$, $\forall n$

(i.e. Countable intersections of open sets need not be open)

Let $I_n = (-\frac{1}{n}, \frac{1}{n})$, I_n^c is closed

$\bigcup_{n=1}^{\infty} I_n^c = \mathbb{R} \setminus \{0\}$ - not closed.

$\frac{1}{n} \to 0 \notin \mathbb{R} \setminus \{0\}$.

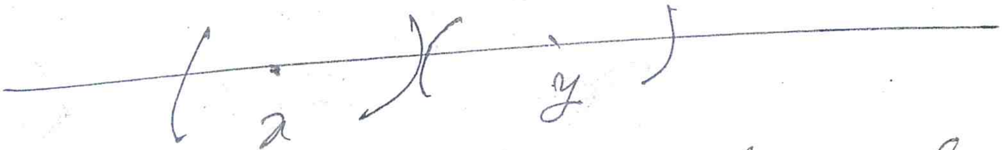
Open set in $\mathbb{R}$:

let $I_i = (a_i, b_i)$, $I_i \cap I_j = \phi$ if $i \neq j$

Then, $\bigcup_{i=1}^{\infty} I_i$ is open.

Result: Any open set $O \subset \mathbb{R}$ is the countable union of open disjoint intervals.

$$x \in O \Rightarrow (x-\epsilon, x+\epsilon) \subset O$$



Choose largest open interval containing x.

$$a_x = \inf\{a < x : (a, x] \subset O\}$$

$$b_x = \sup\{b > x : [x, b) \subset O\}$$

$$x \in (a_x, b_x) = I_x \text{ (Say)}$$

& $y \in (a_y, b_y) = I_y$ (Say)

if $x \neq y$, assume $x < y$

Then by def^n of a_x & b_x,

$$(a_x, b_x) \cap (a_y, b_y) = \Phi$$

Thus $x \in O \Rightarrow x \in I_x = (a_x, b_x) \subset O$

$$\Rightarrow \bigcup_{x \in O} I_x \subset O \subset \bigcup_{x \in O} I_x$$

$$\Rightarrow O = \bigcup_{x \in I_x} I_x$$

Choose $r_x \in I_x \cap Q$; (Q - rationals)

Then $x \to r_x$ is 1-1 map ;

Since $I_x \cap I_y = \Phi$.

$$\Rightarrow O = \bigcup_{r_x \in Q} I_{r_x}$$