Drawback of Riemann Integration

Let $f:[a,b] \longrightarrow \mathbb{R}$ & f is bounded on $[a,b]$.
Then $f \in \mathcal{R}[a,b]$ (i.e. f is Riemann Integrable)
iff f is almost continuous.

However, there are functions which are neither almost cont. nor bounded etc.

(I.) $f:[0,1] \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} 1 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ 0 & x \in \mathbb{Q} \cap [0,1] \end{cases}$

Then $\inf U(P,f) = 1$ & $\sup L(P,f) = 0$.
$\Rightarrow f \notin \mathcal{R}[0,1]$.

(II) $\displaystyle\int_0^1 \frac{1}{\sqrt{t}}\,dt$, $f(t) = \frac{1}{\sqrt{t}}$ is not bounded near "0". However, $\displaystyle\int_{1/n}^1 \frac{1}{\sqrt{t}}\,dt = 2\left(1 - \frac{1}{\sqrt{n}}\right) < 2.$

Question is should we write

$$\int_0^1 \frac{1}{\sqrt{t}}\,dt = \sup_n \int_{1/n}^1 \frac{1}{\sqrt{t}}\,dt = 2 ?$$

(III) $\displaystyle\int_0^\infty \frac{1}{1+t^2}\,dt$, $\displaystyle\int_0^n \frac{1}{1+t^2}\,dt = \tan^{-1} n < \pi/2$

Does it suitable to write

$$\int_0^\infty \frac{1}{1+t^2}\,dt = \sup_n \int_0^n \frac{1}{1+t^2} = \frac{\pi}{2} ?$$

Lebesgue outer measure:

For open interval $I = (a,b)$ assign the length $\ell(I) = b-a$. For $I = (a,\infty)$ or $(-\infty,b)$, we assign $\ell(I) = \infty$.

Now, the question is to assign an appropriate length to an arbitrary subset of $\mathbb{R}$. If $O \subset \mathbb{R}$ is open, then $O = \dot{\bigcup}_{n=1}^{\infty} I_n$, $I_n = (a_n, b_n)$

& $I_n \cap I_m = \phi$ $n \neq m$. In this case, we can consider $\ell(O) = \sum_{n=1}^{\infty} \ell(I_n)$. However, if $A \subseteq \mathbb{R}$, $A \subseteq O \subset \mathbb{R}$. Hence,

$$A \subset \dot{\bigcup_{n=1}^{\infty}} I_n . \text{ Thus, we have}$$

an over-estimate for length of A.

i.e. $\ell(A) \leq \sum \ell(I_n)$, st $A \subset \dot{\bigcup}_{n=1}^{\infty} I_n$

Therefore, we assign a number to A via $m^*(A) := \inf\left\{ \sum \ell(I_n) : A \subset \bigcup I_n \right\}$.

= the outer measure of A.

Notice that, we ~~don't~~ <u>don't</u> require disjointness in of the cover $\{I_n : n \in \mathbb{N}\}$ of A. Moreover, I_n could be any type of interval

e.g. (a_n, b_n) or $[a_n, b_n)$, $[a_n, b_n]$, $(a_n, b_n]$

Since $\Phi \subset (0, \epsilon)$, $\forall \epsilon > 0$, $m^*(\phi) \leq \epsilon$, $\forall \epsilon >$

Hence $m^*(\phi) = 0$. For $a \in \mathbb{R}$,

$\{a\} \subset (a - \epsilon/2, a + \epsilon/2)$.

$\Rightarrow m^*(\{a\}) \leq \epsilon$, $\forall \epsilon > 0$.

$\Rightarrow m^*(\{a\}) = 0$.

Properties of m^*:

(i) If $A \subset B$, then $m^*(A) \leq m^*(B)$.

Let $B \subset \bigcup I_n$, then $A \subset \bigcup I_n$. By def^n, $m^*(A) \leq \sum l(I_n)$; $B \subset \bigcup I_n$.

$\Rightarrow m^*(A) \leq \inf \{ \sum l(I_n) : \bigcup I_n \supset B \}$.

i.e. $m^*(A) \leq m^*(B)$.

(ii) If $\{A_n\}_{n=1}^{\infty}$ is a sequence of subsets in $\mathbb{R}$, then

$$m^*(\bigcup A_n) \leq \sum m^*(A_n).$$ a cover

By def^n of infimum, for $\epsilon > 0$, $\exists$ $\{I_{n,k}\}_{k=1}^{\infty}$ of A_n s.t.

$$\sum_{k=1}^{\infty} l(I_{n,k}) < m^*(A_n) + \epsilon/2^n. \quad (\text{if } m^*(A_n) < \infty)$$

Thus, $\{ I_{n,k} : k = 1, 2, \ldots ; n = 1, 2, \ldots \}$ is a cover of $\bigcup_{n=1}^{\infty} A_n$.

Therefore, $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} l(I_{n,k})$.

$$\leq \sum_{n=1}^{\infty}\left(m^*(A_n) + \frac{\epsilon}{2^n}\right)$$

ie $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum m^*(A_n) + \epsilon, \; \forall \epsilon > 0.$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

Ex. If $A \subset \mathbb{R}$ is countable then

$$A = \{a_1, a_2, \dots\} = \bigcup_{i=1}^{\infty} \{a_i\}$$

$$m^*(A) \leq \sum m^*(\{a_i\}) = 0 \Rightarrow m^*(A) = 0.$$

Thus, $m^*(\mathbb{Q}) = 0$. Alternatively, we can think, $\mathbb{Q} \subset \bigcup_{n \in \mathbb{Z}} \left(x_n - \frac{\epsilon}{2^{|n|+1}}, x_n + \frac{\epsilon}{2^{|n|+1}}\right)$.

$$\Rightarrow m^*(\mathbb{Q}) \leq \sum l\left(x_n - \frac{\epsilon}{2^{|n|+1}}, x_n + \frac{\epsilon}{2^{|n|+1}}\right)$$

$$= \epsilon/2, \; \forall \epsilon > 0.$$

Result: If I ~~is~~ any interval with end pts a & b. Then $m^*(I) = b-a$.

proof: We prove the result for each type of interval. Suppose $I = [a,b]$ and $m^*(I) = b-a$.

Then for $I = (a,b)$, we can deduce (113)

that $[a+\epsilon/2, b-\epsilon/2] \subset (a,b)$

$\therefore m^*([a+\epsilon/2, b-\epsilon/2]) \leq m^*\{(a,b)\}$.

i.e. $b-a \leq m^*\{(a,b)\}$.

Now, (a,b) is a cover of itself.

it sufficient $m^*\{(a,b)\} \leq l\{(a,b)\} = b-a$.

Other interval can be done in similar way. Now, consider the case of ~~proving~~ proving $m^*([a,b]) = b-a$.

$[a,b] \subset (a-\frac{1}{n}, b+\frac{1}{n})$, $\forall n \geq 1$

$m^*([a,b]) \leq b-a+\frac{2}{n} \to b-a$.

On the other hand, suppose

$$[a,b] \subset \bigcup_{n=1}^{\infty} I_n.$$

then $[a,b] \subset \bigcup_{n=1}^{K} I_n$ (Exercise)

(Hint: use Bolzano-Weierstrass theorem).

$$\Rightarrow (a,b) \subset \bigcup_{n=1}^{K} I_n.$$

By induction, $b-a \leq \sum_{n=1}^{K} l(I_n)$.

(if $(a,b) \subset \bigcup_{n=1}^{k} I_n \cup I_{k+1}$. Then $(a,b) \subset \bigcup_{n=1}^{k} I_n$ (114)

or $(a,b) \subset I_{k+1}$. thus,

$$b-a \leq \sum_{n=1}^{k+1} l(I_n).)$$

$\Rightarrow$ $b-a \leq \sum_{n=1}^{\infty} l(I_n)$ for $\{I_n\}_{n=1}^{\infty}$ that cover $[a,b]$. Hence

$$b-a \leq m^*([a,b]) \leq b-a.$$

Ex. let $A \subset \mathbb{R}$ & $x \in \mathbb{R}$. Then for $A+x = \{a+x : a \in A\}$, we have

$$m^*(A+x) = m^*(A).$$

let $A \subset \bigcup I_n$. Then $A+x \subset \bigcup I_n + x = \bigcup(I_n+x)$

i.e. $\{I_n+x\}_{n=1}^{\infty}$ is a cover of $A+x$.

Hence $m^*(A+x) \leq \sum l(I_n+x) = \sum l(I_n)$

$\forall$ cover $\{I_n\}$ of A.

Therefore $m^*(A+x) \leq m^*(A)$. By replace $x \to -x$, $m^*(A-x) \leq m^*(A)$. Replacing A by $A+x$, $m^*(A) \leq m^*(A+x)$.

$$\Rightarrow m^*(A+x) = m^*(A).$$

i.e. m^* is translation invariant.

Result: Let $A \subset \mathbb{R}$ & $\epsilon > 0$. Then $\exists$ an open set $O \supset A$ st $m^*(O) < m^*(A) + \epsilon$. (115)

i.e. $m^*(A) = \inf\{m^*(O) : O \supset A\}$.

Pf: By def^n, for $\epsilon > 0$, $\exists$ $\{I_n\}$ that cover A st $\sum l(I_n) < m^*(A) + \epsilon$ (if $m^*(A) < \infty$.)

But $m^*(\bigcup I_n) \leq \sum l(I_n) < m^*(A) + \epsilon$

Let $O = \bigcup I_n$. Then $m^*(O) < m^*(A) + \epsilon$.

Result: If $A \subset \mathbb{R}$, then $\exists$ a G_δ-set $G \subset \mathbb{R}$ s.t $m^*(A) = m^*(G)$.

Proof: By the previous result for $\epsilon = 1/n$ $\exists$ open set $O_n \supset A$ st

$$m^*(O_n) < m^*(A) + \frac{1}{n}.$$

Let $G = \bigcap O_n$ (a G_δ-set in $\mathbb{R}$).

Then $A \subset G \subset O_n$. Thus

$$m^*(A) \leq m^*(G) \leq m^*(O_n) < m^*(A) + \frac{1}{n}.$$

~~i.e. $m^*(A) < m^*$~~

$$m^*(A) \leq m^*(G) \leq m^*(A) + \frac{1}{n}, \ \forall n \geq 1$$

$$\Rightarrow m^*(A) = m^*(G).$$

Ex. Let $E = \cup E_n$, $E_n \subset \mathbb{R}$. Then $m^*(E) = 0$ iff $m^*(E_n) = 0$, $\forall n \geq 1$.

<u>Solution:</u> $m^*(E) \leq \sum m^*(E_n)$. If each of $m^*(E_n) = 0 \Rightarrow m^*(E) = 0$. (116)

Conversely, suppose $m^*(E) = 0$ & $m^*(E_{n_0}) > 0$. for some $n_0 \in \mathbb{N}$.

Then for $\epsilon = \frac{1}{2} m^*(E_{n_0}) > 0$, $\exists$ a cover $\{I_k\}$ of E st

$$\sum l(I_k) < m^*(E) + \frac{1}{2} m^*(E_{n_0})$$

But $E_{n_0} \subset E \subset \cup I_k \Rightarrow m^*(E_{n_0}) < \sum l(I_k)$

ie $m^*(E_{n_0}) < \frac{1}{2} m^*(E_{n_0})$ X.

Ex. Let $O = \cup I_n$, I_n - open intervals

Then $m^*(O) = \sum \cancel{m} l(I_n)$.

For $\epsilon > 0$, $\exists$ a cover $\{J_k\}$ of O st

$$\sum l(I_k) < m^*(O) + \epsilon \quad \text{---(1)}$$

Now, $\cup I_n = O \subset \cup J_k$. Since I_n's are disjoint, each $I_n \subset J_{k_n}$. ~~for some k_{n_0}~~.

$$l(I_n) \leq l(J_{k_n})$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) < \sum_{n=1}^{\infty} l(J_{K_n}) \leq \sum_{n=1}^{\infty} l(J_K) < m^*(O) + \epsilon$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) < m^*(O) + \epsilon, \quad \forall \epsilon > 0$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \leq m^*(O) \leq \sum l(I_n).$$

ie $m^*(\dot{\bigcup} I_n) = \sum l(I_n) = \sum m^*(I_n)$.

<u>Corollary</u>: If $\{O_i\}_{i=1}^{\infty}$ is a family of disjoint open sets in $\mathbb{R}$, then

$$m^*(\dot{\bigcup} O_i) = \sum m^*(O_i).$$

$$m^*(\dot{\bigcup} O_i) = m^*\left(\dot{\bigcup_{i=1}} \dot{\bigcup_{n=1}^{\infty}} I_{i,n}\right) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} l(I_{i,n}).$$

ie $m^*(\dot{\bigcup}_{i=1}^{\infty} O_i) = \sum_{i=1}^{\infty} m^*(O_i)$.

Question: What are all those sets for which m^* is countably additive. ie

$$m^*(\dot{\bigcup} E_n) = \sum m^*(E_n)\,?$$

Ex. Suppose G is an open & bounded set in $\mathbb{R}$. Then for $\forall \epsilon > 0$, $\exists$ a compact set $K \subset G$ st $m^*(K) > m^*(G) - \epsilon$.

Since G is bounded, $G \subset [\alpha, \beta]$

$\Rightarrow m^*(G) \leq \beta - \alpha < \infty$.

Further, G is open, therefore,

$$G = \bigcup I_n \Rightarrow m^*(G) = \sum l(I_n) < \infty$$

$\therefore$ for $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t.

$$\sum_{n=N+1}^{\infty} l(I_n) < \epsilon/2 \quad \text{———} (1)$$

Let $K = \bigcup_{n=1}^{N} \left[a_n + \frac{\epsilon}{4N}, b_n - \frac{\epsilon}{4N}\right]$, $I_n = (a_n, b_n)$.

Then $m^*(K) = \sum_{n=1}^{N} m^*\left[a_n + \frac{\epsilon}{4N}, b_n - \frac{\epsilon}{4N}\right]$ (notice that prove letter)

$$= \sum_{n=1}^{N}\left(l(I_n) - \frac{\epsilon}{2N}\right) = \sum_{n=1}^{N} l(I_n) - \epsilon/2.$$

$$\therefore m^*(K) = \sum_{n=1}^{N} l(I_n) + \epsilon/2 - \epsilon$$

$$> \sum_{n=1}^{N} l(I_n) + \sum_{n=N+1}^{\infty} l(I_n) - \epsilon$$

$$= m^*(G) - \epsilon.$$

Result: If $[a,b] \cap [c,d] = \emptyset$, then

$$m^*([a,b] \cup [c,d]) = m^*([a,b]) + m^*([c,d])$$

Proof: Since $[a,b] \cap [c,d] = \emptyset$. Then $[a,b]$ & $[c,d]$ will be separated by distance $\epsilon > 0$. (say)

Suppose $[a,b] \cup [c,d] \subset \bigcup I_n$. Then

$$[a,b] \subset \bigcup (I_n \cap (a-\epsilon, b+\epsilon)) = \bigcup I_n' \text{ (say)}$$

& $[c,d] \subset \bigcup (I_n \cap (c-\epsilon, d+\epsilon)) = \bigcup I_n'' \text{ (say)}$

Then $I_n' \cap I_l'' = \emptyset$ $\forall$ $n, l \in \mathbb{N}$.

$$\Rightarrow m^*([a,b]) + m^*([c,d]) \leq \sum l(I_n') + \sum l(I_n'')$$

$$= \sum l(I_n' \cup I_n'')$$

$$= \sum l\{I_n \cap ((a-\epsilon, b+\epsilon) \cup (c-\epsilon, d+\epsilon))\}$$

$$m^*([a,b]) + m^*([c,d]) < \sum_{n=1}^{\infty} l(I_n).$$

$$m^*([a,b]) + m^*([c,d]) \leq m^*([a,b] \cup [c,d]).$$

Since m^* is ~~an~~ sub-additive, other inequality holds.

<u>Observation</u>: If G is an open & bounded subset of $\mathbb{R}$, then for each $\epsilon > 0$, $\exists$ open set O & compact set K s.t.

$$K \subset G \subset O \ \&$$

$$m^*(O) - m^*(K) < \epsilon.$$

In general, we fail to write

$$m^*(B \setminus A) = m^*(B) - m^*(A)$$

for $A \subseteq B$ (we will see example later).

Lebesgue measurable sets:

A set $E \subset \mathbb{R}$ is said to be L-meas. (Lebesgue measurable), if $\forall \epsilon > 0$, $\exists$ open set O & closed sets F s.t

$$F \subset E \subset O \ \& \ m^*(O \setminus F) < \epsilon$$

Note that $m^*(O \setminus E) \leq m^*(O \setminus F) < \epsilon$
and $m^*(F \setminus E) \leq m^*(O \setminus E) < \epsilon$

Thus, we can interpretate that L-measurable sets are approximately open & closed.

<u>Results</u> Let $\mathcal{M}$ denote the class of all L-measurable subsets of $\mathbb{R}$. Then

(i) If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.

$$O^c \subset E^c \subset F^c \ \& \ m^*(F^c \setminus O^c) < \epsilon$$

(ii) If $m^*(E) = 0$. Then $E \in \mathcal{M}$.

For $\epsilon > 0$, $\exists$ $O \supset E$ s.t

$$m^*(O) < 0 + \epsilon$$

Let F be any closed set in E. Then

$$m^*(F) \leq m^*(E) = 0.$$

$\therefore m^*(O \setminus F) \leq m^*(O) < \epsilon$. Thus, $E \in \mathcal{M}$.

(iii) If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$, then $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$.

write $E_n' = E_n \setminus \bigcup_{i=1}^{n-1} E_i$, then $\bigcup E_n' = \bigcup E_n$.
where E_n' are pairwise disjoint sets
(i.e. $E_n' \cap E_m' = \phi$ if $n \neq m$). Thus, w.l.g.
we can assume $E = \bigcup_{n=1}^{\infty} E_n$, $E_n \cap E_m = \phi$.

Suppose $m^*(E) < \infty$, then $m(E_n) \leq m^*(E) < \infty$.
For $\epsilon > 0$, $\exists$ $F_n \subset E_n \subset O_n$ s.t. $m^*(O_n \setminus F_n) < \epsilon/2^n$.

Now,
$$\sum_{n=1}^{K} m^*(O_n) \leq \sum_{n=1}^{K} m^*(O_n \setminus F_n) + \sum_{n=1}^{K} m^*(F_n).$$
$$< \sum_{n=1}^{K} \frac{\epsilon}{2^n} + m^*\left(\bigcup_{n=1}^{K} F_n\right) \quad [\because F_n \text{ is closed \& bounded}]$$
$$< \epsilon + m^*(E) < \infty, \ \forall K \geq 1.$$

i.e. $\sum_{n=1}^{\infty} m^*(O_n) < \infty$. For $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.
$\sum_{n=n_0+1}^{\infty} m^*(O_n) < \epsilon$. Let $\bigcup_{n=1}^{\infty} O_n = O$ & $F = \bigcup_{n=1}^{n_0} F_n$.

Then,
$$m^*(O \setminus F) = m^*\left(\left(\bigcup_{n=1}^{n_0} O_n\right) \cup \left(\bigcup_{n=n_0+1}^{\infty} O_n\right) \setminus \bigcup_{n=1}^{n_0} F_n\right)$$
$$\leq m^*\left(\bigcup_{n=1}^{n_0} (O_n \setminus F_n)\right) + m^*\left(\bigcup_{n=n_0+1}^{\infty} O_n\right)$$
$(\because A \cup B \setminus C = (A \setminus C) \cup (B \setminus C)$
& $F_n \subset E_n \subset O_n)$
$$< \sum_{n=1}^{n_0} \epsilon/2^n + \sum_{n=n_0+1}^{\infty} m^*(O_n) < 2\epsilon.$$

i.e. $F \subset E \subset O$ & $\forall \epsilon > 0$, $m^*(O \setminus F) < 2\epsilon$.
$\Rightarrow E \in \mathcal{M}$.

If $m^*(E) = \infty$, write $E = \bigcup_{k \in \mathbb{Z}} E \cap [k, k+1) = \bigcup_{k \in \mathbb{Z}} A_k$.
and can be done in similar way.

(iv) If $E_1, E_2 \in \mathcal{M}$, then

$$E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1)$$

But for $\epsilon > 0$, $\exists$ $O_i \supset E_i \supset F_i$ s.t.

$$m^*(O_i \setminus F_i) < \epsilon/2 \; ; \; i = 1, 2.$$

For $O = O_1 \cup O_2$, $F = F_1 \cup F_2$

$$O \setminus F \subseteq \bigcup_{i=1}^{2} (O_i \setminus F_i) \Rightarrow m^*(O \setminus F) < \epsilon.$$

$(E_1 \cap E_2)^c = E_1^c \cup E_2^c \in \mathcal{M}$, since $E \in \mathcal{M}$

$\Rightarrow m^*(O \setminus F) < \epsilon$, $O^c \subseteq E^c \subseteq F^c$

$$m^*(F^c \setminus O^c) = m^*(F^c \cap O) < \epsilon.$$

Thus, $\mathcal{M}$ is closed under countable union/intersection & complement.

Note that such family of sets is called σ-algebra.

i.e. if $\mathcal{J} \subset P(\mathbb{R})$ s.t. (i) $A \in \mathcal{J} \Rightarrow A^c \in \mathcal{J}$

(ii) $A_i \in \mathcal{J} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{J}$, is called a σ-algebra of sets.

$\mathcal{B}(\mathbb{R}) = \sigma\{(a,b) : a, b \in \mathbb{R}, b-a < \infty\}$

$\downarrow$ = σ-alg generated by countable

Borel σ-alg) union & complement of sets of type (a,b) & $b-a < \infty$.

Result: Let $a, b \in \mathbb{R}$ & $a < b$, $b - a < \infty$.

Then $I = (a,b) \in \mathcal{M}$.

Proof: For $\epsilon > 0$, $[a+\epsilon, b-\epsilon] \subset (a,b)$ &

$$m^*\{(a,b) \setminus [a+\epsilon, b-\epsilon]\}$$

$$= m^*\{(a, a+\epsilon) \dot{\cup} (b-\epsilon, b)\} \text{ (for small } \epsilon > 0\text{)}$$

$$\leq m^*\{(a, a+\epsilon)\} + m^*\{(b-\epsilon, b)\}$$

$$= 2\epsilon$$

Since I is open, it follows that $(a,b) \in M$.

Now, $[a,b) = \{a\} \cup (a,b)$ & $m^*(\{a\}) = 0$

$\Rightarrow \{a\} \in M$ & $(a,b) \in M$

$\Rightarrow [a,b)$ & $[a,b] \in M$.

Thus, any open set $O = \dot{\cup} I_n \in M$.

Since M is closed under complement any closed set $F \in M$.

~~Ex. If $F \subseteq E$ such $m^*(F) = 0$. Then~~

Ex. If A & $B \subset \mathbb{R}$ st $m^*(A) = 0$.

Then $m^*(A \cup B) = m^*(B)$.

$$m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B) \leq m^*(A \cup B).$$