(123)

Result: Let $x \in \mathbb{R}$ & $E \in \mathcal{M}$. Then $x + E \in \mathcal{M}$.

For $\epsilon > 0$, $\exists$ $F \subset E \subset O$ st
$$m^*(O \setminus F) < \epsilon.$$

But $F + x$ is closed & $O + x = \cup(I_n + x)$ is open with $F + x \subset E + x \subset O + x$

Now, $m^*(O + x \setminus (F + x)) = m^*(O \setminus F) < \epsilon$.

Ex. Verify that

(i) $(F + x)^c = F^c + x$

(ii) $(O + x) \cap (F + x)^c = O \cap F^c + x$.

Hint: $z \notin F + x \Rightarrow z - x \notin F \Rightarrow z - x \in F^c$
$\Rightarrow z \in F^c + x$ etc.

Result: If $E = \cup E_n$, $E_n \in \mathcal{M}$. Then
$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m^*(E_n).$$

(i) Suppose E is bounded, then
$$m^*(E) < \infty \Rightarrow m^*(E_n) < \infty.$$

For $\epsilon > 0$, $\exists$ $F_n \subset E_n \subset O_n$ st
$$m^*(O_n \setminus F_n) < \epsilon/2^n.$$

Now, $\sum_{n=1}^{k} m^*(E_n) \le \sum_{n=1}^{k} \left(m^*(F_n) + m^*(O_n \setminus F_n)\right)$

$$< \sum_{n=1}^{k} m^*(F_n) + \sum_{n=1}^{k} \frac{\epsilon}{2^n} < \sum_{n=1}^{k} m^*(F_n) + \epsilon$$

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$\left(\because\ E_n = (E_n \setminus F_n) \cup F_n \subseteq (O_n \setminus F_n) \cup F_n\right)$

Since F_n's are compact (closed & bounded).

$$\sum_{n=1}^{k} m^*(E_n) < \sum_{n=1}^{k} m^*(F_n) + \epsilon = \left(\bigcup_{n=1}^{k} F_n\right) + \epsilon$$

ie $\sum_{n=1}^{k} m^*(E_n) < m^*(E) + \epsilon \quad \forall\, k \ge 1.$

$$\Rightarrow \sum_{n=1}^{k} m^*(E_n) \le m^*(E) \le \sum m^*(E_n).$$

Now, Suppose, E is not bounded. Then as $\mathbb{R} = \bigcup_{k=1}^{\infty} (k, k+1]$, let

$$A_k = E \cap (k, k+1], \quad E_{n,k} = E_n \cap (k, k+1]$$

Then $E = \bigcup_{k \in \mathbb{Z}} A_k$, $E_n = \bigcup_{k \in \mathbb{Z}} E_{n,k}$.

Now, $\sum_{n=1}^{\infty} m^*(E_n) \le \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} m^*(E_{n,k})$ —— (1)

Since $A_k = \bigcup_{\cancel{k \in \mathbb{Z}}\ n=1}^{\infty} E_{n,k}$, A_k is bounded.

$$m^*(A_k) = \sum_{\cancel{k \in \mathbb{Z}}\ n=1}^{\infty} m^*(E_{n,k}) \quad \text{—— (2)}$$

$\therefore \sum_{n=1}^{\infty} m^*(E_n) \leq \sum_{k=-\infty}^{\infty} m^*(A_k)$ ——— (3)

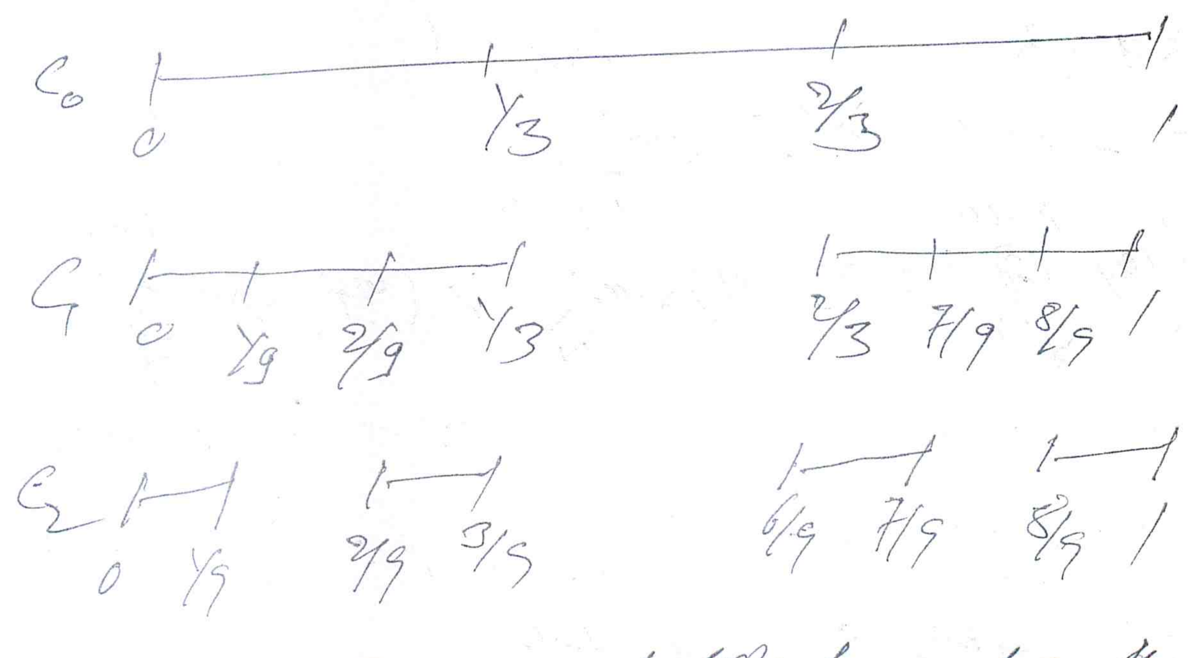
Also $\sum_{k=-l}^{l} m^*(A_k) = m^*\left(\bigcup_{k=-l}^{l} A_k\right) \leq m^*(E), \; \forall\, l \geq 1$

If $m^*(E) = \infty$, the identity holds trivially as $m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n)$. Let $m^*(E) < \infty$,

$\Rightarrow \sum_{k=-\infty}^{\infty} m^*(A_k) \leq m^*(E)$ ——— (4)

$\Rightarrow \sum_{n=1}^{\infty} m^*(E_n) \leq m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

Cantor set: $C_0 = [0,1]$



$C_0 = [0,1]$ 1 interval, length $= 1$.

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ 2 intervals, length $= 1/3$

$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [8/9, 1]$

For C_2, we get 2^2 intervals, each of length $1/3^2$. (126)

By induction C_n consists of 2^n intervals each having ~~3^{-n}~~ length 3^{-n}.

(i) C_n is a decreasing seq^n of closed & bounded sets, thus $C_n \in M$.

(ii) Let $C = \bigcap_{n=1}^{\infty} C_n$, then C contains all the ends pts of the intervals.

(iii) $C = [0,1] \setminus \{ (\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \cup \cdots \}$

(iv) Since $C \subset C_n$, $\forall n \geq 0$.

$$m^*(C) \leq m^*(C_n) = 2^n \cdot \frac{1}{3^n} \to 0.$$

$\therefore m^*(C) = 0$.

(v) C is nowhere dense in $[0,1]$.

i.e. $(\overline{C})^0 = C^0 = \Phi$. If not so, then $C^0 \neq \Phi$ & $x \in C^0$. But C^0 is open $\exists (y,z) \subset C^0 \subset C$, $y < z$.

Thus, $m^*\{(y,z)\} \leq m^*(C) = 0$ X.

(vi) <u>Cantor set is uncountable</u>:

Consider the end pt $\frac{1}{3} \in C$. We can write $\frac{1}{3} = \frac{0}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots \infty = (0.222\cdots)_3$

end pt $x = \frac{2}{3} = (0.2)_3$. Similarly, we will show that each end point can be written as

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots \infty, \quad a_i \in \{0, 2\}.$$

For this, consider the set

$$F = \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} ; a_i \in \{0,1,2\} \right\} \setminus \{\text{end points}\}.$$

For $x \in F$, we have

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots,$$

Notice that $a_1 = 1$ iff $x \in (\frac{1}{3}, \frac{2}{3})$ iff $x \notin C$.

~~If~~ $a_1 \neq 1$, $a_2 = 1$ iff $x \in (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$ iff $x \notin C$

Thus, ~~if~~ $a_i = 1$ for some i iff $x \notin C$.

$$\Rightarrow C = \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} : a_i \in \{0,2\} \right\}.$$

Define $f : C \longrightarrow [0,1]$, by

$$f(x) = f\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i} \right) = \sum_{i=1}^{\infty} \frac{a_i}{2} 2^{-i}.$$

Then $\frac{a_i}{2} \in \{0,1\}$. Thus, $f(x) \in [0,1]$.

<u>f is not 1-1:</u>

$$f(1/3) = f((0.022\cdots)_3) = (0.011\cdots)_2$$
$$= (0.1)_2 = 1/2.$$

(∵ binary repⁿ is not unique)

$$f\left(\tfrac{2}{3}\right) = f((0.2)_3) = (0.1)_2 = \tfrac{1}{2}$$

$$\Rightarrow f\left(\tfrac{1}{3}\right) = f\left(\tfrac{2}{3}\right).$$

Ex. Show that ~~$f(x,y)$~~ $f(x) = f(y)$ iff x, y are end points of G_n's.

<u>f is ~~not~~ onto:</u> $f: C \longrightarrow [0,1] \ni y$ s.t. $f(x) = y$, consider $y = \sum_{j=1}^{\infty} b_j 2^{-j}$, $b_j \in \{0,1\}$. Take $x = \sum 2b_j 3^{-j} \in C$. Then $f(x) = y$ holds. Thus, f is onto.

Therefore, C is uncountable having outer measure zero.

<u>Non-measurable set:</u>

For $x, y \in \mathbb{R}$, define $x \sim y$ iff $x - y \in \mathbb{Q}$. Then $\sim$ is an equivalent relation on $\mathbb{R}$. Hence it partitions the $\mathbb{R}$ into disjoint equivalence class.

Let $x+Q = \{x+r : r \in Q\}$. Then $x+Q$ is an equivalence class under $\sim$.

(i) $(x+Q) \cap [0,1] \neq \Phi$ (easy)

(ii) Let E be a subset of $[0,1]$ that contains exactly one member from each $x+Q$, $x \in \mathbb{R}$.

Let $Q \cap [-1,1] = \{r_1, r_2, \dots\}$ & write $E_i = E + r_i$, $i = 1, 2, \dots$

(iii) $E_i \cap E_j = \Phi$, if $i \neq j$.

if $z \in E_i \cap E_j \Rightarrow z = x + r_i = y + r_j$

$\Rightarrow x - y = r_j - r_i \in Q$

ie. $x \sim y$ Contradiction to the defn of E, as E contains exactly one member each $x+Q$.

(iv) $[0,1] \subset \bigcup_{i=1}^{\infty} E_i \subset [-1,2]$.

Let $x \in [0,1]$. Then $x+Q$ must contains a pt of E. That is, $\exists !\, y \in (x+Q) \cap E$.

$y - x \in Q \cap [-1,1]$.

$y - x = r_{i_0} \Rightarrow x = y - r_{i_0} \in E_{i_0}$.

The set E is not L-measurable.
On contrary if $E \in \mathcal{M}$. Then

$$1 \leq m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq 3$$

$$1 \leq \sum_{i=1}^{\infty} m(E) \leq 3,$$ which is not possible, because $m^*(E) \geq 0$,
if $m^*(E) = 0$, ~~then~~ $m^*(E_i) = 0$. But

$$[0,1] \subset \bigcup E_i$$

$$\Rightarrow \quad 1 \leq \sum m^*(E_i) = 0.$$

Remark: (i) m^* is not countably additive.
Let $A = \bigcup_{i=1}^{\infty} E_i$. Then

$$1 \leq m^*(A) \leq 3.$$

But $\sum_{i=1}^{\infty} m^*(E_i) = \infty$. Thus,

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq 3 < \infty = \sum m^*(E_i).$$

(ii) Whether m^* is finitely additive?
Suppose $m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m^*(A_i)$,

for any $A_1, \ldots, A_n \in P(\mathbb{R})$ = power set of $\mathbb{R}$.
(in other words, let m^* be finitely additive)

Now, $n\, m^*(E) = \sum_{i=1}^{n} m^*(E_i)$

$$= m^*\left(\sum_{i=1}^{n} E_i\right) \le m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le 3.$$

i.e. $m^*(E) \le 3/n, \ \forall n \ge 1.$

$$\Rightarrow m^*(E) = 0 \quad \times$$

$\therefore m^*$ can not be finitely additive.

(iii) Suppose $A \subset E$ & $A \in \mathcal{M}$,

then $m^*(A) = 0$.

For this, let $A_i = A + r_i$, $r_i \in \mathbb{Q} \cap [-1,1]$.

Then $\bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{\infty} E_i \subset [-1,2]$.

Since A is L-measurable, each of $A_i \in \mathcal{M}$. Thus

$$m^*\left(\bigcup_{i=1}^{n} A_i\right) \le 3$$

$$\sum_{i=1}^{n} m^*(A_i) \le 3$$

$n\, m^*(A) \le 3 \Rightarrow m^*(A) \le \frac{3}{n}, \forall n.$

$$\Rightarrow m^*(A) = 0.$$