Result if f, g are L-m'sble, then

$$\max\{f, g\} = \frac{f+g+|f-g|}{2}$$

$$\min\{f, g\} = \frac{f+g-|f-g|}{2}$$

$\inf f_n$, $\sup f_n$, $\overline{\lim} f_n$, $\underline{\lim} f_n$, $\lim f_n$ all are L-m'sble.

Result: If $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be such that $f(x) \neq 0$, $\forall x \in \mathbb{R}$, then $\frac{1}{f}$ is m'sble.

Proof:

$$\left\{x : \frac{1}{f(x)} > \alpha\right\} = \underbrace{\left\{x : f(x) > \frac{1}{\alpha}\right\}}_{\phi \text{ if } \alpha < 0} \cup \underbrace{\left\{x : f(x) < \frac{1}{\alpha}\right\}}_{\phi \text{ if } \alpha > 0}$$

$\in \mathcal{M}$.

Simple function:

Let $E_i \in \mathcal{M}$ & $\alpha_i \in \mathbb{R}$. Then

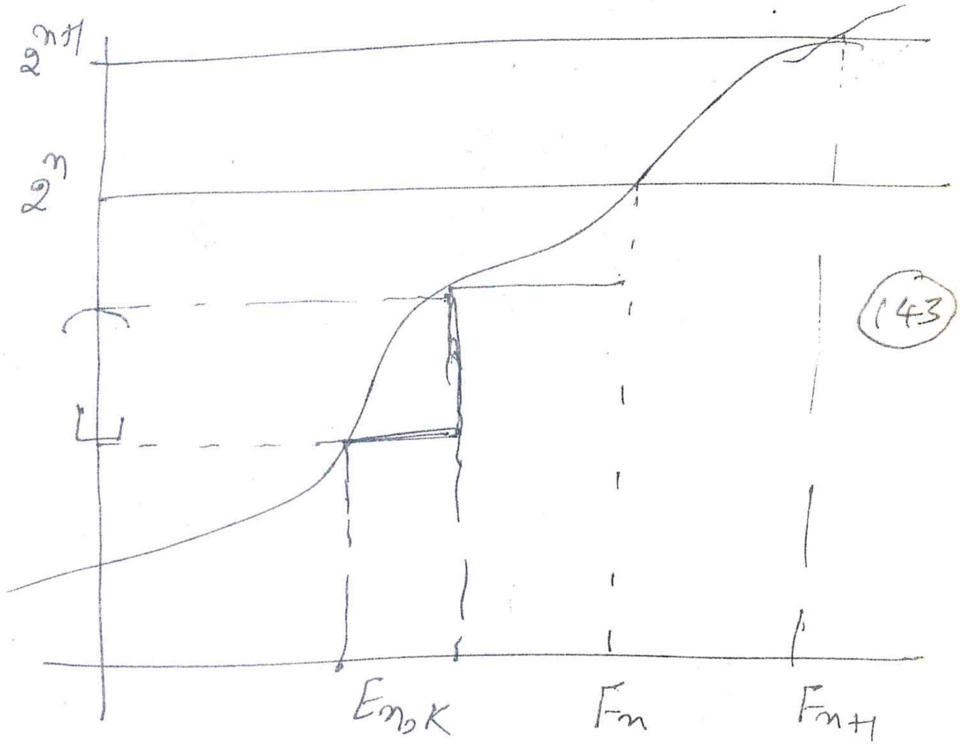
$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$ is called a simple function.

Ex. $\varphi = 1 \cdot \chi_{[0,1]} + 2 \cdot \chi_{[9,3]}$

Theorem: Let $f : \mathbb{R}[0,\infty] \longrightarrow \overline{\mathbb{R}}[0,\infty]$ be a m'sble function. Then $\exists$ a sequence φ_n of simple functions s.t. (i) $\varphi_n \uparrow$ & $\varphi_n \leq f$ (ii) $\varphi_n \to f$ point wise, (iii) $\varphi_n \to f$ uniformly on any set A where f is bounded.

Proof: We 1st devide the image of f in $[0, 2^n]$ into 2^{2n} disjoint parts.

$k = 0, 1, 2, \dots, 2^{2n} - 1$



$$f^{-1}\left\{\left[\frac{K}{2^n}, \frac{K+1}{2^n}\right)\right\} = E_{n,K} \quad \& \quad f^{-1}\{[2^n, \infty]\} = F_n.$$

Then (i) $\varphi_n \geq 0$, (ii) $E_{n,K}$'s are disjoint m'ble sets.

(iii) $\varphi_n \uparrow$ on $[0, \infty]$. claim $\varphi_n(x) \leq \varphi_{n+1}(x)$.

If $x \in E_{n,K} = \left\{x : \frac{2K}{2^{n+1}} \leq f(x) < \frac{2K+2}{2^{n+1}}\right\}$

$$= E_{n+1, 2K} \,\dot{\cup}\, E_{n+1, 2K+1}.$$

For $x \in E_{n+1, 2K}$, $\varphi_n(x) = \frac{K}{2^n} = \frac{2K}{2^{n+1}} = \varphi_{n+1}(x)$.

For $x \in E_{n+1, 2K+1}$, $\varphi_n(x) \leq \frac{2K+1}{2^{n+1}} = \varphi_{n+1}(x)$.

If $x \in F_n$, then $x \in (F_n \setminus F_{n+1}) \,\dot{\cup}\, F_{n+1}$.

For $x \in F_{n+1}$, $\varphi_n(x) = 2^n < 2^{n+1} = \varphi_{n+1}(x)$.

For $x \in F_n \setminus F_{n+1}$, we have

$$2^n = \frac{2^{2n+1}}{2^{n+1}} \leq f(x) < 2^{n+1} = \frac{2^{2n+2}}{2^{n+1}}$$

i.e. $x \in E_{n+1, 2^{2n+1}} \cup \dots \cup E_{n+1, 2^{2n+2}-1}$

Then $\varphi_{n+1}(x) \in \left\{ \frac{2^{2n+1}}{2^{n+1}}, \dots, \frac{2^{2n+2}-1}{2^{n+1}} \right\}$. Thus,

$\varphi_n(x) = 2^n = \frac{2^{2n+1}}{2^{n+1}} \leq \varphi_{n+1}(x)$. That is,

$\varphi_n \uparrow$ & $\varphi_n \leq f$.

(iv) $\varphi_n \to f$ pointwise.

Let $f(x) < \infty$. Then

$\{x : f(x) < \infty\} = \bigcup_{n=1}^{\infty} \{x : f(x) < 2^n\}$,

$\Rightarrow f(x) < 2^n$ for some n, and hence

$x \in E_{n,k} \Rightarrow \varphi_n(x) = \frac{k}{2^n}$

$\therefore \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \Rightarrow 0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}$, $\forall n \geq 1$.

$\Rightarrow \varphi_n \to f$ pointwise. ——— $(*)$

(~~B~~) ~~$\varphi_n \to f$~~ If $f(x) = \infty$, for some x.

Then $\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \geq 2^n\}$.

i.e. ~~$f(x) \geq 2^n$~~ $\varphi_n(x) = 2^n \to \infty = f(x)$.

(v) $\varphi_n \to f$ uniformly on a set when f is bounded. Let $E = \{x : f(x) \leq M\}$.

Then $\exists\, n_0$ s.t. $f(x) < 2^n$, $\forall n \geq n_0$.

Hence from $(*)$ $0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}$, $\forall n \geq n_0$

Notice that n_0 is free (or unique for E) of $x \in E$. Thus,

$$0 \le \sup\left(f(x) - \varphi_n(x)\right) \le \frac{1}{2^n} \to 0.$$

Hence $\varphi_n \to f$ uniformly on E. (145)

Cor: If $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ is m'ble. Then $\exists$ seq^n of simple function s.t.

(i) $|\varphi_n| \uparrow |f|$ point-wise and

Proof: $f = f^+ - f^-$. Then $\exists$ $\varphi_n^+ \uparrow f^+$ and $\varphi_n^- \uparrow f^-$. that is

$$\varphi_n = \varphi_n^+ - \varphi_n^- \longrightarrow f^+ - f^- = f.$$

$$|\varphi_n| = \varphi_n^+ + \varphi_n^- \le f^+ + f^- \;\; \& \; |\varphi_n| \uparrow |f|.$$

In this case, $|f - \varphi_n| = |f^+ - \varphi_n^+ + f^- - \varphi_n^-| \to 0$.

& $\varphi_n \to f$ uniformly on $E = \{x : |f(x)| < M\}$

Note that, $f^+ = \max\{f, 0\}$ &

~~$f^- = -\min\{f(x), 0\}$.~~

$f^- = -\min\{f, 0\}$.

Adoptations: $0 \cdot \infty = 0$, $\infty \cdot 0 = 0$

so $0 \cdot m(\mathbb{R}) = 0$, $\infty \cdot m(\mathbb{Q}) = 0$.

a Notations: $\frac{\infty}{\infty}$, $\infty - \infty$.

Lebesgue Integration:

Let $\varphi : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ s.t

$$\varphi = \sum_{j=1}^{n} \alpha_j \chi_{E_j}, \quad \alpha_j \in [0,\infty],$$

& $E_j \in \mathcal{M}$ & $m(E_j) \le \infty$. Then we write

$$\int_{\mathbb{R}} \varphi \, dm = \sum_{j=1}^{n} \alpha_j \, m(E_j).$$

Remark 1. $\int_{\mathbb{R}} \varphi \, dm = 0$ iff $\varphi = 0$.

Now, if $E \in \mathcal{M}$, then $\varphi|_E = \sum_{j=1}^{n} \alpha_j \chi_{E_j \cap E}$,

hence $\int_E \varphi \, dm = \sum_{j=1}^{n} \alpha_j \, m(E_j \cap E)$.

Notice that $(\mathbb{R}, \mathcal{M}, m)$ is called Lebesgue measure space. If $E \in \mathcal{M}$, then for

$\mathcal{M}_E = \{F \cap E : F \in \mathcal{M}\}$, $(E, \mathcal{M}_E, m)$

is also a ~~also a~~ L-measure space on E.

Remark 2: Since $E_1 \cup E_2 = (E_1 \setminus E_2) \,\dot\cup\, (E_1 \cap E_2) \,\dot\cup\, (E_2 \setminus E_1)$

in the def^n of φ, we can assume $\{E_j : j = 1, 2, \ldots, n\}$ is a disjoint family.

i.e. $E_j \cap E_i = \phi$ if $i \ne j$.

Now, let $f : \mathbb{R} \xrightarrow{\text{m'ble}} [0,\infty]$, then $\exists$ a seq^n of simple functions $\varphi_n \uparrow f$ point wise. Hence $\int_{\mathbb{R}} \varphi_n \, dm \uparrow$ sequence in $\overline{\mathbb{R}}$.

(147)

Thus, we define

$$\int_{\mathbb{R}} f \, dm := \lim_{n \to \infty} \int_{\mathbb{R}} \varphi_n \, dm$$

or $$\int_{\mathbb{R}} f \, dm = \sup \left\{ \int_{\mathbb{R}} \varphi \, dm : \varphi \le f \right\}.$$

If $f : \mathbb{R} \xrightarrow{\text{m'ble}} \overline{\mathbb{R}}$, then $f = f^+ - f^-$, we write $\int_{\mathbb{R}} f \, dm = \int_{\mathbb{R}} f^+ \, dm - \int_{\mathbb{R}} f^- \, dm$, if at least either of $\int_{\mathbb{R}} f^+ \, dm$ or $\int_{\mathbb{R}} f^- \, dm$ is finite.

Let $\mathcal{L}^+(\mathbb{R}, \mathcal{M}, m) = \left\{ f : \mathbb{R} \xrightarrow{\text{m'ble}} [0,\infty] \right\}$.

<u>Result</u>: For φ, ψ simple functions in $\mathcal{L}^+(\mathbb{R}, \mathcal{M}, m)$, & $c \in \mathbb{R} = [0,\infty)$,

(i) $\int_{\mathbb{R}} c\varphi = c \int_{\mathbb{R}} \varphi$

(ii) $\int_{\mathbb{R}} (\varphi + \psi) = \int_{\mathbb{R}} \varphi + \int_{\mathbb{R}} \psi$.

(iii) If $\varphi \le \psi$, then $\int_{\mathbb{R}} \varphi \, dm \le \int_{\mathbb{R}} \psi \, dm$.

Proof: (i) is trivial.

(ii) Let $\varphi = \sum_{j=1}^{m} \alpha_j E_j$, $\psi = \sum_{k=1}^{m} \beta_k F_k$.

Notice that by assigning 0 on $\left(\bigcup_{j=1}^{m} E_j\right)^c$, we can assume that $\mathbb{R} = \bigcup_{j=1}^{m} E_j$, $\mathbb{R} = \bigcup_{k=1}^{m} F_k$.

Then $E_j = \bigcup_{k=1}^{m} (E_j \cap F_k)$, $F_k = \bigcup_{j=1}^{m} (E_j \cup F_k)$.

Now,

$$\int_{\mathbb{R}} \varphi \, dm + \int_{\mathbb{R}} \psi \, dm = \sum_{j=1}^{m} \sum_{k=1}^{m} \alpha_j \, m(E_j \cap F_k) + \sum_{k=1}^{m} \sum_{j=1}^{m} \beta_k \, m(E_j \cap F_k)$$

$$= \sum_{k=1}^{m} \sum_{j=1}^{m} (\alpha_j + \beta_k) \, m(E_j \cap F_k) \quad \text{— (1)}$$

$$\int_{\mathbb{R}} (\varphi + \psi) \, dm = \int_{\mathbb{R}} \sum \sum (\alpha_j + \beta_k) \, m(E_j \cap F_k)$$

$$= \int_{\mathbb{R}} \varphi \, dm + \int_{\mathbb{R}} \psi \, dm, \quad (\text{by } (1)).$$

(iii) If $\varphi \le \psi$, then $\alpha_j \le \beta_k$, when $E_j \cap F_k \ne \Phi$.

$$\int_{\mathbb{R}} \varphi \, dm = \sum_{j=1}^{m} \sum_{k=1}^{m} \alpha_j \, m(E_j \cap F_k) \le \sum_{j=1}^{m} \sum_{k=1}^{m} \beta_k \, m(E_j \cap F_k)$$

$$= \int_{\mathbb{R}} \psi \, dm.$$

Result: If $f, g \in L^+(\mathbb{R}, \mathcal{M}, m)$, then for $f \le g$, $\int_{\mathbb{R}} f \, dm \le \int_{\mathbb{R}} g \, dm$.

For this, let $\varphi \leq f$, φ is simple, then

$$\varphi \leq g \Rightarrow \int_{\mathbb{R}} f\,dm = \sup_{\varphi \leq f} \int_{\mathbb{R}} \varphi \leq \int_{\mathbb{R}} g\,dm.$$

Result: If $f, g \in L^+(\mathbb{R}, \mathcal{M}, m)$, then (14-9)

$$\int (f+g)\,dm = \int f\,dm + \int g\,dm.$$

(we prove it later !)

Monotone Convergence theorem (MCT)

Let $f, f_n \in L^+(\mathbb{R}, \mathcal{M}, m)$ be such that $f_n \uparrow f$ point-wise, then

$$\int_{\mathbb{R}} f\,dm = \lim \int_{\mathbb{R}} f_n\,dm.$$

Since $f_n \leq f \Rightarrow \int f_n \leq \int f$

$$\Rightarrow \lim_{n\to\infty} \int f_n \leq \int f \quad (\because \int f_n \uparrow)$$

Note that proof of other inequality is bit involved and hence we omit it here

Remark 1. $f_n \uparrow f$ is necessary in MCT.

e.g. $f_n = \frac{1}{n} \chi_{[0,n]} \to 0.$

$$\int_{\mathbb{R}} f_n\,dm = 1 \neq 0 = \int \lim f_n.$$

Ex. Verify MCT for $f_n : \mathbb{R} \longrightarrow [0,\infty]$, given by (i) $f_n = \chi_{(n, n+1)}$

(ii) $f_n = n\chi_{(0, \frac{1}{n})}$.

Remark: Integration is a linear map on $L^+(\mathbb{R}, M, m)$, i.e. $f \longmapsto \int_{\mathbb{R}} f\,dm$ is linear.

Let $f, g \in L^+(\mathbb{R}, M, m)$. Then $\exists\ \varphi_n \uparrow f$ & $\psi_n \uparrow g$. By MCT,

$$\int_{\mathbb{R}} (f+g)\,dm = \lim \int_{\mathbb{R}} (\varphi_n + \psi_n) = \lim \int_{\mathbb{R}} \varphi_n + \lim \int_{\mathbb{R}} \psi_n$$

$$= \int_{\mathbb{R}} f\,dm + \int_{\mathbb{R}} g\,dm.$$

Ex. For $E \in M$, and $f \in L^+(\mathbb{R}, M, m)$, if $\int_E f\,dm = 0$, then $f = 0$, ~~Since~~ provided $m(E) > 0$.

$$\int_E f\,dm = \sup_{\varphi \le f} \int_E \varphi = 0 \Rightarrow \int_E \varphi = 0 \Rightarrow \varphi = 0.$$

Cor to MCT: Let $f, f_n \in L^+(\mathbb{R}, M, m)$ be such that $f_n \uparrow f$ p.w. a.e. on $\mathbb{R}$. Then

$$\int_{\mathbb{R}} f\,dm = \lim \int_{\mathbb{R}} f_n.$$

Proof: Let $f_n \xrightarrow{p.w.} f$ on A, then $m^*(A^c) = 0$.

$\Rightarrow A, A^c \in M$. That is,

$$\chi_E f_n \longrightarrow \chi_E f$$

By MCT, $\int \chi_A f = \lim \int \chi_A f_n$

$$\Rightarrow \int_A f = \lim \int_A f_n$$

now, $\int_{\mathbb{R}} f = \int_A f + \int_{A^c} f = \lim \int_A f_n + \lim \int_{A^c} f_n$

or $\int_{\mathbb{R}} f = \lim \int_{\mathbb{R}} f_n$.

<u>Theorem</u>: Let $f \in L^+(\mathbb{R}, \mathcal{M}, m)$. Then

$$\int_{\mathbb{R}} f \, dm = 0 \text{ iff } f = 0 \text{ a.e. } \mathbb{R}.$$

Proof. For $f = \varphi = \sum_{j=1}^{m} \alpha_j \chi_{E_j}$, $\int_{\mathbb{R}} \varphi \, dm = 0$ iff

either $\alpha_j = 0$ or $m(E_j) = 0$, $\forall j = 1, 2, \ldots, n$

ie. $\int_{\mathbb{R}} \varphi \, dm = 0$ iff $\varphi = 0$ a.e.

Now, if $f = 0$ a.e., $\int_{\mathbb{R}} f \, dm = \sup_{\varphi \le f} \int \varphi \, dm = 0$. (by previous ca

~~Conversely~~, Suppose $\int_{\mathbb{R}} f \, dm = 0$. Then

Consider $E = \{x \in \mathbb{R} : f(x) > 0\}$ ~~$= A \{x : f(x)$~~

$$= \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : f(x) > \tfrac{1}{n}\} = \bigcup_{n=1}^{\infty} E_n. \text{ (say)}$$

Now, $m(E_n) = n \int_{E_n} \frac{1}{n} \, dm \le n \int_{E_n} f \, dm \le n \int_{\mathbb{R}} f \, dm = 0$

$$\Rightarrow m(E) = 0 \Rightarrow f = 0 \text{ a.e.}$$

Fatou's lemma:

let $f_n \in L^+(\mathbb{R}, \mathcal{M}, m)$. Then

$$\int_{\mathbb{R}} \underline{\lim}\, f_n \, dm \leq \underline{\lim} \int_{\mathbb{R}} f_n \, dm.$$

Proof: let $g_k = \inf_{n \geq k} f_n$. Then $g_k \leq f_j,\ \forall j \geq k$

Thus, $\int g_k \leq \inf_{j \geq k} \int f_j$. ~~$\forall j \geq k$~~. Now,

$g_k \uparrow \sup_{k \geq 1} \left(\inf_{n \geq k} f_n \right)$. By MCT,

$$\int_{\mathbb{R}} \underline{\lim}\, f_n = \int \lim_{k \to \infty} g_k = \lim_{k \to \infty} \int g_k \leq \lim_{k \to \infty} \inf_{j \geq k} \int f_n$$

Remark 1. Strict inequality can hold.

For $f_n = \frac{1}{n} \chi_{[0,n]} \xrightarrow{\text{unif}} 0$,

$$\int_{\mathbb{R}} \underline{\lim}\, f_n = 0 < 1 = \underline{\lim} \int_{\mathbb{R}} f_n.$$

Remark 2: Fatou's lemma need not hold beyond non-negative functions.

Ex. let $f_n = -\frac{1}{n} \chi_{[n, 2n]}$, $\forall n \geq 1$.

$$\inf_{n \geq k} f_n(x) = \inf_{n \geq k} \left\{ -\frac{1}{n} \right\} = -\frac{1}{k}$$

$\sup_{k \geq 1} \left(\inf_{n \geq k} f_n(x) \right) = 0$ ie $\underline{\lim}\, f_n(x) = 0$.

$$\int_{\mathbb{R}} \underline{\lim}\, f_n = 0 > -1 = \underline{\lim} \int_{\mathbb{R}} f_n.$$

Let $f : (\mathbb{R}, \mathcal{M}, m) \xrightarrow{\text{m'ble}} \overline{\mathbb{R}} = [-\infty, \infty]$.

Then $f = f^+ - f^-$, & f^+, f^- are L-m'ble. (153)

Defn: If $\int_{\mathbb{R}} f^+ < \infty$ & $\int_{\mathbb{R}} f^- < \infty$ both holds, then we say f is <u>integrable</u> and $\int_{\mathbb{R}} f\, dm = \int_{\mathbb{R}} f^+\, dm - \int_{\mathbb{R}} f^-\, dm$.

Since $|f| = f^+ + f^-$, it follows that $\int_{\mathbb{R}} f\, dm$ is finite iff $\int_{\mathbb{R}} |f|\, dm$ is finite.

Let $L^1(\mathbb{R}, \mathcal{M}, m) = \{ f : \mathbb{R} \xrightarrow{\text{m'ble}} \overline{\mathbb{R}} \quad \int_{\mathbb{R}} |f| < \infty \}$.

We also use the symbols $L^1(\mathbb{R})$ & $L^1(\mathbb{R}, m)$ & $L^1(\mathbb{R}, \mathcal{M}, m)$.

Notice that L^1 is a linear space on $\mathbb{R}$.

Since $\int |f| = 0$ iff $|f| = 0$ a.e.
iff $f = 0$ a.e.

If we adopt, ~~$\int |f|$~~ $f = 0$ iff $f = 0$ a.e.

Then $L^1(\mathbb{R}, \mathcal{M}, m)$ is a normed linear space with $\|f\|_1 = \int |f|$.

Result: If $f \in L^1(\mathbb{R}, \mathcal{M}, m)$, then

$$\left|\int_{\mathbb{R}} f \, dm\right| \leq \int_{\mathbb{R}} |f| \, dm.$$

Proof: $$\left|\int_{\mathbb{R}} f \, dm\right| = \left|\int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-\right| \leq \left|\int_{\mathbb{R}} f^+\right| + \left|\int_{\mathbb{R}} f^-\right|$$
$$= \int_{\mathbb{R}} f^+ + \int_{\mathbb{R}} f^- = \int_{\mathbb{R}} |f|.$$

Chebyshev's inequality:

Let $f \in L^1(\mathbb{R}, \mathcal{M}, m)$. Then

$$m\{x \in \mathbb{R} : |f(x)| \geq \alpha\} \leq \frac{1}{\alpha} \|f\|_1.$$

Proof: $$\text{LHS} = \frac{1}{\alpha}\int_{\{x : |f(x)| \geq \alpha\}} \alpha \, dm \leq \frac{1}{\alpha}\int_{\{x : |f(x)| \geq \alpha\}} |f(x)| \, dm \leq \frac{1}{\alpha}\int_{\mathbb{R}} |f| \, dm$$

Cor: If $f \in L^1(\mathbb{R}, \mathcal{M}, m)$, then

$$m\{x \in \mathbb{R} : |f(x)| = \infty\} = 0.$$

i.e. an L^1-function is almost finite.

$$m\{x : |f(x)| = \infty\} = m\left\{\bigcap \{x : |f(x)| \geq n\}\right\}$$

But $m\{x : |f(x)| \geq n\} \leq \frac{1}{n} \|f\|_1$.

$\therefore$ $m\{x : |f(x)| = \infty\} \leq m\{x : |f(x)| \geq n\} \leq \frac{1}{n} \|f\|_1$
$\to 0$ as $n \to \infty$.

Dominated Convergence theorem (DCT):

Let $f_n : (\mathbb{R}, \mathcal{M}, m) \xrightarrow{m'ble} \overline{\mathbb{R}}$ seq^n s.t.

(i) $f_n(x) \longrightarrow f(x)$ (pointwise), $\forall x \in \mathbb{R}$.

(ii) $|f_n| \leq g \in L^1(\mathbb{R}, \mathcal{M}, m)$. (155)

Then $\int_{\mathbb{R}} f \, dm = \lim \int_{\mathbb{R}} f_n \, dm$.

Proof: Since $f_n \xrightarrow{p.w.} f$ & $|f_n| \leq g \in L^1(\mathbb{R}, \mathcal{M}, m)$.

$\Rightarrow |f_n| \to |f| \Rightarrow |f| \leq g \in L^1 \Rightarrow f \in L^1$.

Now, $0 \leq g + f_n \longrightarrow g + f$ } pointwise

$0 \leq g - f_n \longrightarrow g - f_n$

By Fatou's lemma,

$$\int_{\mathbb{R}} (g+f) = \int_{\mathbb{R}} \underline{\lim}\,(g+f_n) \leq \underline{\lim}\left(\int_{\mathbb{R}} (g+f_n)\right)$$

$$\Rightarrow \int_{\mathbb{R}} f \leq \underline{\lim} \int_{\mathbb{R}} f_n \qquad (\because \textstyle\int_{\mathbb{R}} g < \infty)$$

Similarly, $\int_{\mathbb{R}} (g - f_n) \leq \underline{\lim}\left(\int_{\mathbb{R}} g - \int_{\mathbb{R}} f_n\right)$

$$\cancel{\Rightarrow \underline{\lim} \int_{\mathbb{R}} f_n \geq \int_{\mathbb{R}} f}$$

$$-\int_{\mathbb{R}} f \leq -\overline{\lim} \int_{\mathbb{R}} f_n$$

i.e. $\int_{\mathbb{R}} f \geq \overline{\lim} \int_{\mathbb{R}} f_n$

$$\Rightarrow \overline{\lim} \int_{\mathbb{R}} f_n \leq \int_{\mathbb{R}} f \leq \underline{\lim} \int_{\mathbb{R}} f_n.$$

$$\Rightarrow \lim \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f.$$

Ex. Verify DCT for $f_n : (\mathbb{R}, \mathcal{M}, m) \longrightarrow \mathbb{R}$, where

(i) $f_n = n\,\chi_{[0, \frac{1}{n}]}$

(ii) $f_n = \frac{1}{n}\,\chi_{[n, n+1]}$

(iii) $f_n = \chi_{[n, n+1]}$.

(Hint: $f_n \to 0$, $\int_{\mathbb{R}} f_n = 1$)

Bounded Convergence theorem (BCT).

Let $E \in \mathcal{M}$ & $0 < m(E) < \infty$. If

$$f, f_n : (E, \mathcal{M}_E, m) \longrightarrow \overline{\mathbb{R}}$$

be such that (i) $|f_n(x)| \le M$, $\forall n \ge 1$, $\forall x \in E$.

(ii) $f_n \xrightarrow{p.w.} f$. Then

$$\int_E f = \lim \int_E f_n.$$

Proof: $\int_E |f_n| \le \int_E M = M\,m(E) < \infty$.

$\Rightarrow f_n$ is dominated by M.

By DCT, $\int_E f = \lim \int_E f_n$.

Theorem: If f is bounded. Then $f \in R[a,b]$ iff f is continuous on $[a,b]$ a.e. (157)

i.e. $\exists\ g : [a,b] \xrightarrow{\text{cont}} \mathbb{R}$ st $f = g$ a.e.

Theorem: Every R-integrable function is Lebesgue integrable.

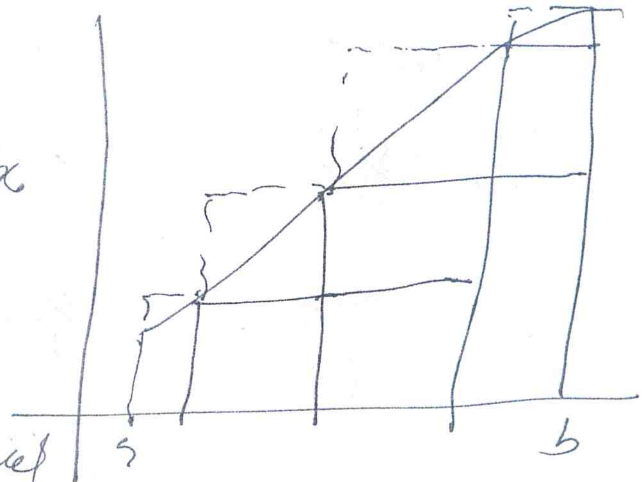
i.e. $R[a,b] \subset L^1[a,b]$.

(i) $f \in R[a,b] \Rightarrow f = g$ a.e., where g is cont on $[a,b]$. Therefore g is m'ble and hence f is m'ble.

If $f \in R[a,b]$, then

$$\inf_P U(P,f) = \int_a^b f(x)\,dx$$

$$\sup_P L(P,f) = \int_a^b f(x)\,dx$$



~~bothex~~ exist & equal to $\int_a^b f(x)$. But for L-integration, we only want $\sup_P L(P,f) = \int f\,dm$.

Hence $f \in R[a,b] \Rightarrow f \in L^1[a,b]$.

(Note that is just an intuition & not a proof.

Theorem: Let $f \in R[a,b]$. Then $f \in L^1[a,b]$, & $\int_{[a,b]} f\,dm = \int_a^b f(x)\,dx$.

Proof: Let $I = [a,b]$ & $f \in R(I)$, then $\exists$ an increasing seq^n of partitions P_n of I s.t. $\lim U(P_n, f) = \lim L(P_n, f) = \int_a^b f(x)\,dx$.

For a partition P of $[a,b]$, denote

$$\varphi_P = \sum_{j=1}^{k} M_j \chi_{(t_{j-1}, t_j]}, \quad M_j = \sup_{[t_{j-1}, t_j]} f(x).$$

$$\& \quad \psi_P = \sum_{j=1}^{k} m_j \chi_{(t_{j-1}, t_j]}, \quad m_j = \inf_{[t_{j-1}, t_j]} f(x)$$

$$P = \{a = t_0 < t_1 < \cdots < t_{j-1} < t_j < \cdots t_k = b\}.$$

Then $\varphi_P \downarrow$ seq^n & $\psi_P \uparrow$ seq^n.

Since $f \in R(I)$, $\exists$ $M, m > 0$ s.t. $m \leq f(x) \leq M$ but then

$$m \leq \psi_{P_n}(x) \leq f(x) \leq \varphi_{P_n}(x) \leq M. \quad \text{———} (1)$$

For each fixed $x \in I$, $\varphi_{P_n}(x) \downarrow$ seq^n bounded below by m & $\psi_{P_n}(x) \uparrow$ bounded above by M, let $\lim \psi_{P_n}(x) = \psi(x)$

$$\& \lim \varphi_{P_n}(x) = \varphi(x).$$

Then $m \le \varphi(x) \le f(x) \le \phi(x) \le M$ —— (2)

Then ψ & φ being limit of simple f^ns, are mble.

(159)

By BCT (Bounded Conv. Thm)

$$\int_I \varphi\, dm = \lim \int_I \varphi_n\, dm = \lim U(P_n, f) = \int_a^b f\, dx.$$

Similarly, $\int_I \psi\, dm = \lim \int_I \psi_n\, dm = \lim L(P_n, f) = \int_a^b f(x)\, dx.$

$$\Rightarrow \int_I (\varphi - \psi)\, dm = 0 \text{ iff } \varphi - \psi = 0 \text{ a.e.}$$

$$(\because \varphi - \psi \ge 0).$$

from $\psi(x) \le f(x) \le \psi(x)$ a.e.

$f(x) = \psi(x)$ a.e. $\Rightarrow$ f is mble

Thus, $\int_{[a,b]} f\, dm = \int_I \psi\, dm = \int_a^b f(x)\, dx.$

i.e. $R[a,b] \subsetneq L^1[a,b]$. Since

$$f = \chi_{(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]}, \quad \int_{[0,1]} f\, dm = 1$$

but $L(P,f) = 0$ & $U(P,f) = 1$, $\forall P$.

— the end —