Open / closed sets in $\mathbb{R}^n$:

For $r>0$, $x \in \mathbb{R}^n$, $x=(x_1,\dots,x_n)$

(i) $B_r(x) = \{y \in \mathbb{R}^n : \|x-y\| < r\}$
is called open ball in $\mathbb{R}^n$

(ii) $\overline{B_r(x)} = \{y \in \mathbb{R}^n : \|x-y\| \le r\}$
is called closed ball in $\mathbb{R}^n$.

(iii) $S_r(x) = \{y \in \mathbb{R}^n : \|x-y\| = r\}$
is called sphere (disc) in $\mathbb{R}^n$.

Defn: A set $O \subset \mathbb{R}^n$ is said to be open if for any $x \in O$, $\exists\, r>0$ s.t. $B_r(x) \subset O$.

Ex. $\{(x,y) : |x|+|y| < 1\}$ is open.

Ex $\{(x,y) : |x| \le 1 \;\&\; |y| < 1\}$ is not open.

Defn: A set $F \subset \mathbb{R}^n$ is said to be closed if its complement $\mathbb{R}^n \setminus F$ (or F^c) is open.

Recall that $(x^k) = (x_1^k, \dots, x_n^k) \in \mathbb{R}^n$ is said to be convergent if $\exists\, x \in \mathbb{R}^n$

such that for any $\epsilon > 0$, $\exists\, K_0 \in \mathbb{N}$ s.t. (18)

$$\|x^k - x\| < \epsilon, \quad \forall\, k \geq k_0$$

Since $\left(\sum_{j=1}^{n} |x_j^k - x_j|^2\right)^{1/2} < \epsilon$, it follows that

$$x^k \to x \text{ iff } x_j^k \to x_j, \quad \forall j = 1, 2, \ldots, n.$$

<u>Result</u>: A set $F \subset \mathbb{R}^n$ is closed if $\forall$ seq^{ns} $(x^k) \subset F$ such that $x^k \to x$, implies $x \in F$.

(Essentially another def^n of closed set).

<u>Result</u>: Both the def^{ns} of closed sets are equivalent.

(I) F is closed if F^c is open

(II) F is closed if $\forall\, x^k \in F$ & $x^k \to x$, implies $x \in F$.

<u>proof</u>: (I) $\Rightarrow$ II :

Let $x^k \in F$ & $x^k \to x$ but

$x \notin F$. Then $x \in F^c$; since F^c is open (by I), $\exists\ r > 0$ st (19)

$$B_r(x) \subset F^c.$$

But for $\epsilon = r > 0$, $\exists\ k_0 \in \mathbb{N}$ st

$$\|x^k - x\| < r, \quad \forall\ k \geq k_0.$$

$$\Rightarrow \cancel{x}\ x^k \in B_r(x) \subset F^c, \quad \forall\ k \geq k_0 \quad \times$$

But notice that $x^k \in F$, $\forall\ k \geq 1$.

Thus. $x \in F$.

<u>(II) $\Rightarrow$ (I):</u> Claim F^c is open.

If <u>not</u>, then $\exists\ x \in F^c$ s.t

$$B_{1/k}(x) \not\subseteq F^c, \quad \forall\ k \geq 1.$$

$$\Rightarrow \exists\ x^k \in B_{1/k}(x), \text{ s.t. } x^k \in F, \quad \forall\ k \geq 1$$

$$\Rightarrow \quad \|x^k - x\| < \frac{1}{k}, \quad \forall\ k \geq 1$$

$$\Rightarrow x^k \to x \ \&\ F \text{ is closed}$$

$$\Rightarrow x \in F. \quad \times$$

Thus, for any $x \in F^c$, $\exists\ k \in \mathbb{N}$ st

$$B_{1/k}(x) \subset F^c.$$

Interior of Set in R^n:

Interior of a set A is the largest open set A^0 contained in A.

i.e. O is open & $O \subset A$, then

$$O \subseteq A^0.$$

Ex. $N^0 = \Phi$, $Q^0 = \Phi$, $(R \setminus Q)^0 = \Phi$.

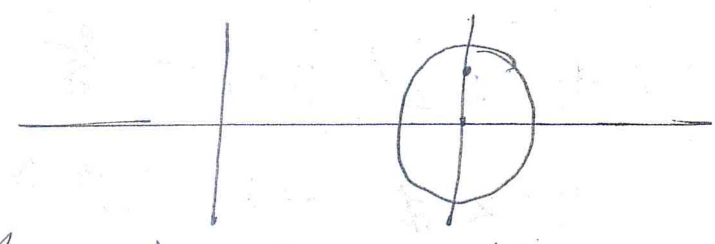
$\{(x,y) : |x| \le 1 \ \& \ |y| < 1\}^0 = \{(x,y) : |x| < 1 \ \& \ |y| < 1\}$.

Ex. The set $\{(x,y) : y = \sin\frac{1}{x},\ x \neq 0\} = A$ is neither open nor closed set in R^2.

If $x_n = \frac{1}{n\pi} \neq 0$, $(x_n, y_n) = (\frac{1}{n\pi}, 0) \in A$, but $\lim (x_n, y_n) = (0,0) \notin A$.

Since any ball

$B_{1/n}(\frac{1}{\pi}, 0) \not\subseteq A$,



$x = 1/\pi$

$\Rightarrow A$ is not open in R^2.

A set $A \subset \mathbb{R}^n$ is said to be bounded if $\exists$ $M > 0$ s.t. $\|x\| \leq M$, $\forall x \in A$. (21)

i.e. $x \in B_M(0)$, $\forall x \in A$

i.e. $A \subset B_M(0)$.

Equivalently, if $A \subset B_M(x_0)$, for some $x_0 \in \mathbb{R}^n$, & $M > 0$. Then we say A is ~~a~~ bounded subset of $\mathbb{R}^n$.

__Closure of a set:__

The closure of a set $A \subset \mathbb{R}^n$ is the smallest closed set $\bar{A}$ that contains A. i.e. If B is closed & $A \subset B$, then $\bar{A} \subset B$.

Ex. $A = \{(x,y) : |x| < 1, |y| < 1\}$, then
$\bar{A} = \{(x,y) : |x| \leq 1, |y| \leq 1\}$.

$A = \{(x,y) : y = \sin\frac{1}{x}, x \neq 0\}$. Then
$\bar{A} = \{(x,y) : y = \sin\frac{1}{x}, x \neq 0\} \cup (\{0\} \times [-1,1])$.

Def^n: A closed and bound subset $K \subset \mathbb{R}^n$ is called a compact subset of $\mathbb{R}^n$.

Ex The set $\{(x,y): y = \sin\frac{1}{x}, x \neq 0\} \cup \{\{0\} \times [-1,1]\}$ is closed but not bounded. as $\mathbb{R} \times \{0\}$ is contained in it.

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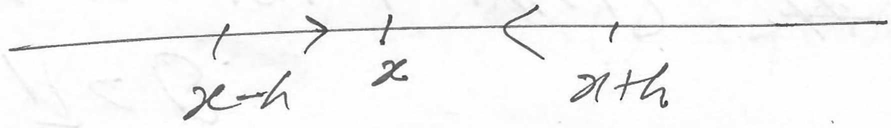
limit at a pt :

Suppose $f : (a,b) \subset \mathbb{R} \longrightarrow \mathbb{R}$



If $\lim\limits_{h\to 0} f(x-h)$ & $\lim\limits_{h\to 0} f(x+h)$ both exist & equal then we say limit a x exists.

$(a,b) \times (c,d)$

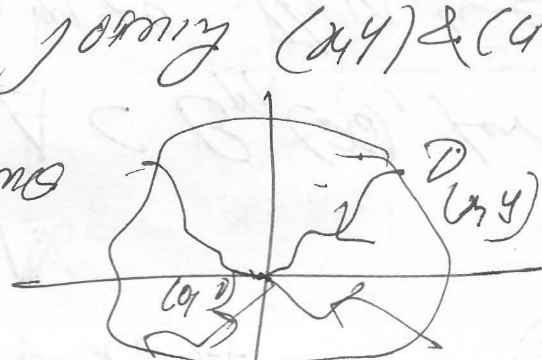
Suppose $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ = finite & equal along all path joining (x,y) & $(0,0)$

let $x = r\cos\theta$, $y = r\sin\theta$

$(x,y) \to (0,0)$ iff

$r^2 = x^2+y^2 \to 0$



i.e. $r^2 \to 0$ as $r \to 0$ $(\because r>0)$.

$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \text{finite}$, i.e.

limit at $(0,0)$ exists. (23)

Say

Let $D = (a_1, b_1) \times \cdots \times (a_n, b_n)$, and

$$f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$f(X) = f(x_1, x_2, \ldots, x_n) = (f_1(x_1, \ldots x_n), \ldots, f_m(x_1, x_2, \ldots$

Then f is said to be continuous at $X \in D$, if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ st

$Y \in D$, $\|X - Y\| < \delta \Rightarrow \|f(X) - f(Y)\| < \epsilon$

$$\left(\sum_{i=1}^{m} |f_i(X) - f_i(Y)|^2\right)^{1/2} < \epsilon$$

$$\Rightarrow |f_i(X) - f_i(Y)| < \epsilon, \ \forall\, i = 1, 2, \ldots$$

i.e. f is cont at $X \Rightarrow$ each of component f_i is cont at X.

Conversely, if each of f_i $(i = 1, 2, \ldots, m)$ is cont. then for $\epsilon > 0$, $\exists\, \delta > 0$ st

$$\|X - Y\| < \delta \Rightarrow |f_i(X) - f_i(Y)| < \frac{\epsilon}{\sqrt{m}}$$

ie $\|X-Y\|<\delta \Rightarrow \|f(X)-f(Y)\|<\epsilon$.

Thus, it is enough to consider

$f: \mathbb{R}^2 \to \mathbb{R}$ for question/result

regarding $f: \mathbb{R}^n \to \mathbb{R}^m$!

Def^n: Let $D \subseteq \mathbb{R}^2$ and $f: D \subset \mathbb{R}^2 \to \mathbb{R}$.

Then f is said to be cont at $X_0 = (x_0, y_0) \in D$ if, $\forall\, \epsilon>0$, $\exists\, \delta>0$ st

$$\|X-X_0\|<\delta,\ X\in D,\ \|f(X)-f(X_0)\|<\epsilon.$$

ie $\displaystyle\lim_{X\to X_0} f(X) = f(X_0)$.

Negation of Continuity:

$\exists\, \epsilon_0>0$ st $\forall\, \delta>0$, $\exists\, X\in D$ st

$$\|X-X_0\|<\delta \text{ but } \|f(X)-f(X_0)\|\geq \epsilon_0$$

Result: $f: D\subset \mathbb{R}^2 \to \mathbb{R}$ is cont at X_0 iff $\forall$ seq^n $X_n \to X_0$, implies

$$f(X_n)\to f(X_0).$$

Proof: $X_0 = (x_0, y_0)$, $X_n = (x_n, y_n)$.

Suppose f is cont at X_0. Then for each $\epsilon > 0$, $\exists\ \delta > 0$ st

$$\|X - X_0\| < \delta \Rightarrow |f(X) - f(X_0)| < \epsilon. \quad -(1)$$

Let $X_n \to X_0$. Then for $\delta > 0$, $\exists\ n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow \|X_n - X_0\| < \delta, \Rightarrow |f(X_n) - f(X_0)| < \epsilon, \text{ (by (1))} \quad -(2)$$

Thus, $X_n \to X_0 \Rightarrow f(X_n) \to f(X_0)$.

Conversely, suppose (2) holds, but f is not cont at X_0, then $\exists\ \epsilon_0 > 0$ st $\forall\ \delta > 0$, $\exists\ X \in D$ s.t. $\|X - X_0\| < \delta$ but $|f(X) - f(X_0)| \geq \epsilon_0$

Take $\delta = \frac{1}{n} > 0$, then $\exists\ X_n \in D$ st

$$\|X_n - X_0\| < 1/n, \text{ but } |f(X_n) - f(X_0)| \geq \epsilon_0.$$

ie $X_n \to X_0$ but $f(X_n) \not\to f(X_0)$.

Ex. $f(x,y) = \begin{cases} 1 & \text{if } xy \neq 0 \\ 0 & \text{o.w.} \end{cases}$

Then $\lim_{(x,y) \to (0,0)} f(x,y)$ does not exists.

But $xy \neq 0$ replaced by $xy = 1$, it exists