Ex. Let $f : \mathbb{R}^2 \to \mathbb{R}$. ~~be a~~ Check the continuity of f at $(0,0)$. (26)

(i) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } x^2+y^2 \neq 0 \\ & \text{o.w.} \end{cases}$

(ii) $f(x,y) = \dfrac{\sin^2(x-y)}{\sqrt{x^2+y^2}}$, $f(0,0) = 0$

(iii) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y} & \text{if } x^2+y \neq 0 \\ & \text{o.w.} \end{cases}$

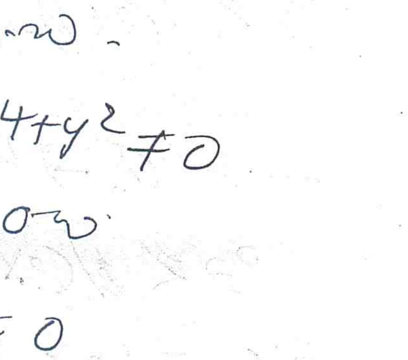
(iv) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if } x^4+y^2 \neq 0 \\ & \text{o.w.} \end{cases}$

(v) $f(x,y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 0 & \text{o.w.} \end{cases}$

ϵ & δ defn:

Let $f(x,y) = \dfrac{xy}{x^2+y^2}$, $f(0,0) = 0$.

For $x = y$, $f(x,x) = 1$



$|f(x,x) - f(0,0)| = 1$

Take $\epsilon = 1/2$, then $\nexists\ \delta > 0$

$\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)| < 1/2$.

Composition of two Continuous f's.

Let $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ & $g : I \subset \mathbb{R} \to \mathbb{R}$ be cont, then $g \circ f$ is cont.

Pf: Since f is cont at $x \in D$, for $\epsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\|x - y\| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \quad -(1)$$

Since g is cont at $f(x)$, for $\eta > 0$, $\exists$ $\mu > 0$ s.t.

$$|f(x) - f(y)| < \mu \Rightarrow |g(f(x)) - g(f(y))| < \eta. \quad -(2)$$

Given $\forall \epsilon \in (0, \infty)$, choose $\epsilon = \mu$. Then from (1),

$$\|x - y\| < \delta \Rightarrow |g\circ f(x) - g\circ f(y)| < \eta.$$

Thus $g \circ f$ is cont at x.

Alt. let $x_n \to x$. Then $f(x_n) \to f(x)$ and hence $g(f(x_n)) \to g(f(x))$.

Ex. $f(x,y) = \begin{cases} \dfrac{\sin xy}{xy}, & \text{if } xy \neq 0 \\ 1 & \text{o.w.} \end{cases}$

$f(x,y) = p \circ g(x,y)$, where $p(t) = \begin{cases} \dfrac{\sin t}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$

$g(x,y) = xy$

Suppose $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is cont (where $D = (a,b) \times (c,d)$ or an open set).

For $x \in D$ & $\epsilon > 0$, $\exists$ $\delta > 0$ st

$$\|x - y\| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

$$y \in B_\delta(x) \Rightarrow f(y) \subset B_\epsilon(f(x))$$

ie $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. ——— $(*)$

<u>Result</u>: $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is cont on D iff for each open set $O \subset \mathbb{R}$, $f^{-1}(O)$ is open in D.

<u>Pf</u>: Suppose f is cont and $O \subset \mathbb{R}$ is open. Then for $x \in f^{-1}(O)$, $f(x) \subset O$. Since O is open $\exists$ $\epsilon > 0$ st $B_\epsilon(f(x)) \subset O$.

Since f is cont at x, $\exists$ $\delta > 0$ st

$$f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset O.$$

$$\Rightarrow B_\delta(x) \subset f^{-1}(O).$$

Hence $f^{-1}(O)$ is open.

Conversely, Suppose $f^{-1}(O)$ is open $\forall$ open set $O \subset \mathbb{R}$. ~~Then~~ Let $\epsilon > 0$.

Then $B_\epsilon(f(x))$ is open in $\mathbb{R}$.

~~The~~ Thus $f^{-1}(B_\epsilon(f(x)))$ is open in D. Since $x \in f^{-1}(B_\epsilon(f(x)))$.

$\exists\ \delta > 0$ st $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$.

Thus, f is cont at x.

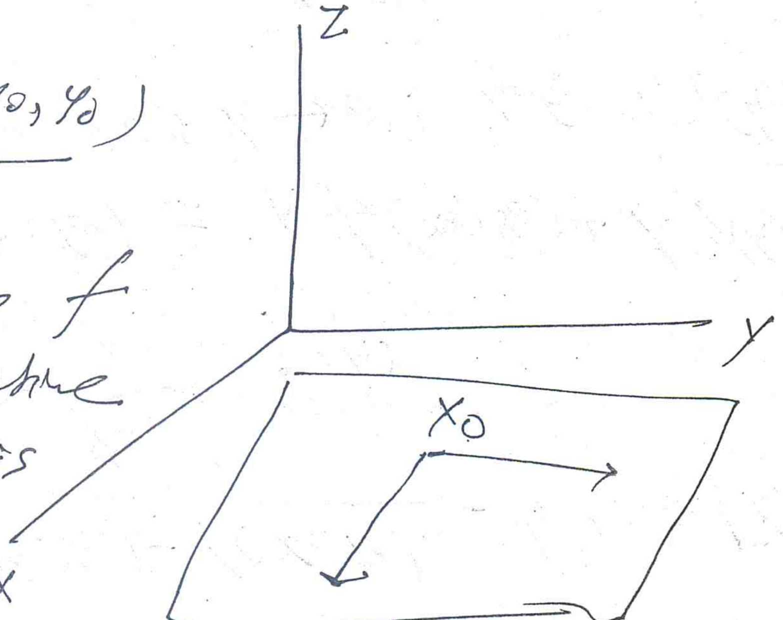
<u>Partial derivatives:</u>

Let $D = (a,b) \times (c,d)$ (or in general open set in $\mathbb{R}^2$). Let $f : D \longrightarrow \mathbb{R}$.

$X_0 = (x_0, y_0)$.

If $\lim\limits_{h \to 0} \dfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$

exists, we say f has partial derivative parallel to x-axis & we denote it by $\dfrac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$.



In other words. for $\epsilon > 0$, $\exists$ $\delta > 0$ st (31)

$$|h| < \delta \Rightarrow \Big| \underbrace{\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} - f_x(x_0, y_0)}_{\eta(h)} \Big| < \epsilon$$

$$f(x_0+h, y_0) - f(x_0, y_0) = h f_x(x_0, y_0) + h\,\eta(h)$$

where $\eta(h) \to 0$ as $h \to 0$. Let $h\eta(h) = \eta'(h)$

$$f(x_0+h, y_0) - f(x_0, y_0) = h f_x(x_0, y_0) + \eta'(h)$$

where $\eta'(h) \to 0$ as $h \to 0$.

Similarly,

$$f(x_0, y_0+k) - f(x_0, y_0) = k f_y(x_0, y_0) + \gamma(k)$$

where $\gamma(k) \to 0$ as $k \to 0$.

Note: From the graph, it is clear that ~~we on~~ existence of partial derivative parallel to x-axis only demands continuity on a small open line segment and need not on nbhd of (x_0, y_0).

Ex. $f(x, y) = \dfrac{xy}{x^2+y^2}$, $f(0,0) = 0$.

Then $f_x(0,0) = 0 = f_y(0,0)$

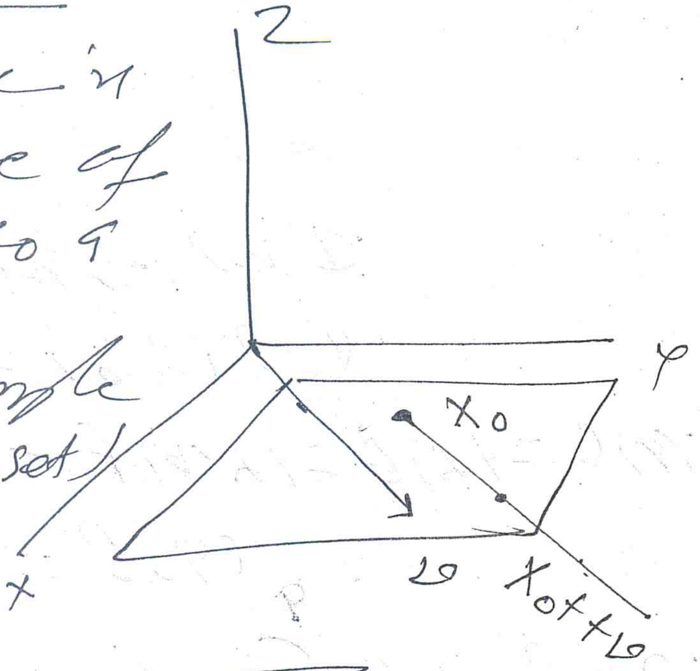
But f is not cont at $(0,0)$.

Directional derivatives (D.D.)

Directional derivative is the rate of change of a function parallel to a given direction.

Let $X_0 \in D$ (rectangle or open set)

& $f: D \subset \mathbb{R}^2 \to \mathbb{R}$



Let $v = (v_1, v_2)$, $\|v\| = \sqrt{v_1^2 + v_2^2} = 1$.

Then D.D. of f at X_0 along v is defined by

$$D_v f(X_0) = \lim_{t \to 0} \frac{f(X_0 + tv) - f(X_0)}{t}$$

Note: f having D.D. at X_0 along v demands ~~on~~ cont only on a line segment parallel to v (and need not in an open nbd of X_0).

Ex. $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } x^4 + y^2 \neq 0 \\ 0 & \text{o.w.} \end{cases}$

$$D_v f(0,0) = \lim_{t \to 0} \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2} = \begin{cases} 0 & v_2 = 0 \\ \dfrac{v_1^2}{v_2} & v_2 \neq 0 \end{cases}$$

But f is not cont at $(0,0)$. for $y = mx$ etc.

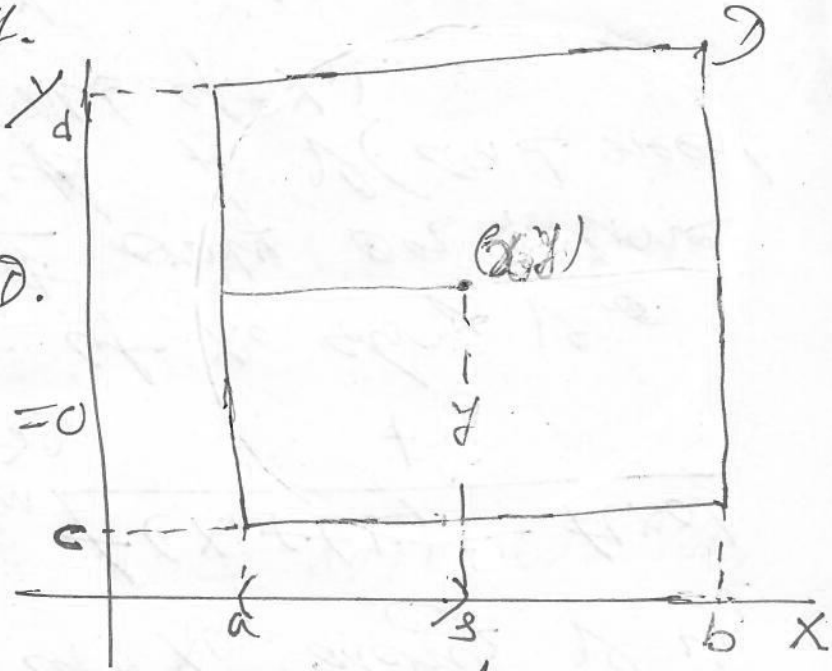
Ex. Let $D=(a,b)\times(c,d)$ (or λ open Convex set in $\mathbb{R}^2$).

(i.e. $\forall\, x,y\in D \Rightarrow \lambda x+(1-\lambda)y\in D,\ \forall\, \lambda\in[0,1]$). (32)

Suppose $f: D\subset\mathbb{R}^2 \longrightarrow \mathbb{R}$ s.t.

$$f_x(x,y)=0=f_y(x,y),\ \forall\, x,y\in D.$$

Then f is const.



Since D is Convex

$$(a,s)\times\{y\}\in D.$$

Thus, $\displaystyle\int_a^s f_x(x,y)\,dx=0$

$$f(s,y)=f(a,y),$$

Let $g(y)=f(a,y)$. Then

$$0=f_y(s,y)=g'(y)\Rightarrow \int_c^t g'(y)\,dy=0.$$

$$\Rightarrow g(y)=g(c).$$

Thus, $f(s,y)=f(a,y)=g(y)=g(c)$.

$$\forall\,(s,y)\in D.$$

$\Rightarrow$ f is const on D.

Remark: A similar prove will work for D is open & Convex.

Ex. Let D be an open set in $\mathbb{R}^2$ and $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be s.t. f_x & f_y both are bounded on D. Then f is cont.

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$$f(x_0+h, y_0+k) - f(x_0, y_0)$$
$$= f(x_0+h, y_0+k) - f(x_0, y_0+k) + f(x_0, y_0+k) - f(x_0, y_0)$$
$$= h f_x(x_0+\theta_1 h, y_0+k) + k f_y(x_0, y_0+\theta_2 k).$$

(by MVT of one-variable)

where $\theta_1, \theta_2 \in (0,1)$.

Hence,

$$|f(x_0+h, y_0+k) - f(x_0,y_0)| \le \sqrt{h^2+k^2}\sqrt{M_1^2+M_2^2}$$

where $|f_x(x,y)| \le M_1$, $|f_y(x,y)| \le M_2$ $\forall (x,y) \in D$.

Thus, $|f(x_0+h, y_0+k) - f(x_0,y_0)| \to 0$ as $\sqrt{h^2+k^2} \to 0$.

$\Rightarrow f$ is cont at (x_0, y_0).

... to check cont. of f at (x_0, y_0), we

Differentiation :

Let D be an open set in $\mathbb{R}^2$.

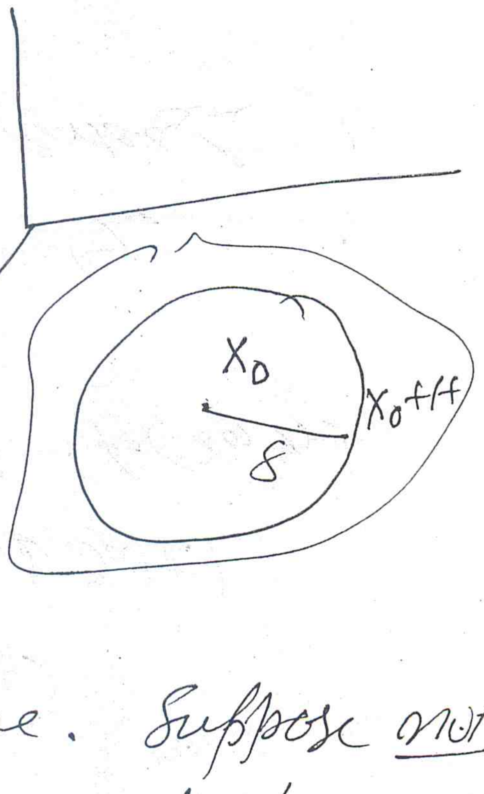
$H = (h, k)$, $X_0 = (x_0, y_0)$.

Then f is said to be diff. at $X_0 \in D$ if $\exists$ $L \in \mathbb{R}^2$ s.t

$$(*)\quad E_L(H) = \frac{f(X_0+H) - f(X_0) - L \cdot H}{\|H\|} \to 0$$

as $\|H\| \to 0$.

Notice that, since we need limit in $(*)$ exists in a δ-nbd of X_0, it means f is diff along all direction including parallel to X-axis & Y-axis.



The vector L is unique. Suppose not, then $\exists$ $M \in \mathbb{R}^2$ st $(*)$ holds.

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Thus, $\frac{(L-M)\cdot H}{\|H\|} = \epsilon_L(H) - \epsilon_M(H) \to 0$ as $\|H\| \to 0$.

Set $H = tv$, $v(\neq 0) \in \mathbb{R}^2$.

Then $\lim_{t\to 0} \frac{|t|\,|(L-M)\cdot v|}{|t|\,\|v\|} = 0 \Rightarrow |(L-M)\cdot v| = 0$ $\forall v \in \mathbb{R}^2$

Consider $v = L-M$. Then

$\|L-M\| = 0$ iff $L = M$.

Hence derivative of f is unique and we write $L = f'(x_0)$.

Since

$$\epsilon(H) = \frac{f(x_0+H) - f(x_0) - H\cdot f'(x_0)}{\|H\|} \to 0$$

as $\|H\| \to 0$.

~~Take~~ Set $H = tv$, $\|v\| = 1$.

$$\epsilon(tv) = \frac{f(x_0+tv) - f(x_0) - t\,v\cdot f'(x_0)}{|t|}$$

$\to 0$ as $t \to 0$.

Thus, $v\cdot f'(x_0) = D_v f(x_0)$.

Put ~~v~~ $v = (1,0)$, then

$D_v f(x_0) = f_x(x_0)$.

Similarly, $v = (0,1)$, $D_v f(x_0) = f_y(x_0)$.