<u>Result:</u> Let $A \subset (X, d)$. Then $x \in \bar{A}$ iff $B_\epsilon(x) \cap A \neq \Phi$, $\forall\ \epsilon > 0$.

pf: Let $x \in \bar{A}$. If $\exists\ \epsilon_0$ s.t. $B_{\epsilon_0}(x) \cap A = \Phi$

then $A \subset (B_{\epsilon_0}(x))^c$ = closed set

By def^n of $\bar{A}$. $\bar{A}$ is the smallest closed set containing A.

$\Rightarrow \quad \bar{A} \subset (B_{\epsilon_0}(x))^c$

~~But~~ $x \in \bar{A}$, but $x \notin (B_{\epsilon_0}(x))^c$.

Conversely, suppose $B_\epsilon(x) \cap A \neq \Phi$ $\forall \epsilon > 0$. Then $B_\epsilon(x) \cap \bar{A} \neq \Phi$.

By previous theorem $x \in \bar{A}$.

<u>Result:</u> $x \in \bar{A}$ iff $\exists$ a seq^n $x_n \in A$ s.t. $x_n \to x$.

If $x \in \bar{A} \Rightarrow B_{1/n}(x) \cap A \neq \Phi$, $\forall n \geq 1$ $\left(\epsilon = \frac{1}{n}\right)$

$\Rightarrow x_n \in B_{1/n}(x)$ & $x_n \in A$.

$d(x_n, x) < \frac{1}{n}$, $\forall n \geq 1$

$\Rightarrow x_n \to x$.

Conversely, if $\exists$ $x_n \in A$ st $x_n \to x$.
then for $\epsilon > 0$, $d(x_n, x) < \epsilon$, $\forall n \geq N$

$$x_n \in B_\epsilon(x) \cap A, \ \forall n \geq N$$

$$\Rightarrow B_\epsilon(x) \cap A \neq \Phi, \ \forall \epsilon > 0$$

$$\Rightarrow x \in \overline{A}.$$

Complete metric spaces:

A metric space (X, d) is said to be complete, if every Cauchy seq^n in X has limit in X.

(i.e. x_n is C.C $\Rightarrow$ $x_n \to x \in X$.)

Ex. $(\mathbb{R}, u)$ is complete.
Let $(x_n) \in \mathbb{R}$ be C.C. Then (x_n) is bdd in $\mathbb{R}$. By Bolzano-Weierstrass theorem, $\exists$ subseq^n $x_{n_k} \to x \in \mathbb{R}$. Then $x_n \to x \in \mathbb{R}$.

Ex. $(\mathbb{R}^n, \|\cdot\|_p)$ is complete for $1 \leq p \leq \infty$.

Let $1 \leq p < \infty$ and $x^k = (x_1^k, x_2^k, \ldots, x_n^k)$ be a C.C. in $(\mathbb{R}^n, \|\cdot\|_p)$. Then

$$\|x^k - x^l\|_p = \left(\sum_{j=1}^{n} |x_j^k - x_j^l|^p\right)^{1/p} < \epsilon, \ \forall l, k \geq N.$$

$\Rightarrow |x_j^l - x_j^k| < \epsilon, \quad \forall\ l, k \geq N.$

i.e. $(x_j^l)_{l=1}^{\infty}$ is a C.S. Hence

$$x_j^l \to x_j \in \mathbb{R}, \quad j = 1, 2, \dots, n.$$

Letting $x = (x_1, x_2, \dots x_n)$. Then

$$\|x^k - x\|_p < n^{\frac{1}{p}} \epsilon, \quad \forall\ k \geq N.$$

$$\Rightarrow x^k \to x \in \mathbb{R}^n.$$

For $p = \infty$, $\|x^k - x^l\|_\infty < \epsilon, \ \forall\ l, k \geq N$

$$\Rightarrow |x_j^k - x_j^l| < \epsilon \text{ etc.}$$

(Similar argument as above).

Ex. Let $1 \leq p \leq \infty$. Then $(\ell^p, \|\cdot\|_p)$ is complete.

Let $1 \leq p < \infty$. ~~Then~~ Let $x^k = (x_1^k, x_2^k, \dots x_n^k, \dots)$ be C.S. ~~Then~~ Then for $\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$. s.t. $\|x^k - x^l\|_p < \epsilon. \quad \forall\ k, l \geq N.$

$$\sum_{j=1}^{n} |x_j^k - x_j^l|^p < \sum_{j=1}^{\infty} |x_j^k - x_j^l|^p < \epsilon^p \to (1)$$

Then for each fixed n, it reduces to $(\mathbb{R}^n, \|\cdot\|_p)$, which is complete.

ie. for fixed n, $\sum_{j=1}^{n} |x_j^k - x_j^l|^p < \epsilon^p$ $\forall$ k, l.

Hence $x_j^k \to x_j$. ~~Let~~ Thus, letting

$k \to \infty$ in (1), implies,

$$\sum_{j=1}^{n} |x_j - x_j^l|^p \le \epsilon^p, \forall l \ge N. \quad \text{(2)}$$

But LHS is an increasing seq^n of n which is bounded above. Thus, by letting $n \to \infty$, we get

$$\sum_{j=1}^{\infty} |x_j - x_j^l|^p \le \epsilon^p$$

ie $\quad \|x - x^l\|_p \le \epsilon, \ \forall l \ge N.$

$$\Rightarrow \|x - x^N\|_p < \epsilon.$$

$$\|x\|_p \le \|x - x^N\|_p + \|x^N\|_p < \epsilon + \|x^N\|_p.$$

$\Rightarrow x \in l^p$.

For $p = \infty$, $\sup_{1 \le j \le n} |x_j^l - x_j^k| \le \sup_{j \in \mathbb{N}} |x_j^l - x_j^k| < \epsilon$

Since $(\mathbb{R}^n, \|\cdot\|_\infty)$ is complete, similar argument will give the result.

~~Result: If $A \subset (X, d)$~~

Result: Every closed subset of a complete metric space is complete.

P: Let F be a closed subset of a complete metric space (X,d). (93)

$\Rightarrow (F,d)$ is a metric space

Let x_n be a C.S. in F. Then x_n is a C.S. in X (which is complete).

$\Rightarrow x_n \to x \in X$, but F is closed

$\Rightarrow x \in F$.

Conversely, if (F,d) is complete, then (F,d) is closed. Since, $x_n \in F$ & $x_n \to x$. Then (x_n) is C.S. in F and $x \in F$. Thus, F is closed.

Notice that here we do not require completeness of X.

Ex. $(c_0, \|\cdot\|_\infty)$ is a closed proper subspace of $(l^\infty, \|\cdot\|_\infty)$.

Let $x^k = (x_1^k, x_2^k, \ldots x_n^k, \ldots) \in c_0$ &

$x^k \to x$. That is $\sup\limits_{j\in\mathbb{N}} |x_j^k - x_j| < \frac{\epsilon}{2}, \forall k \geq N.$

Claim $x \in c_0$. We have

~~Then~~ $|x_j^N - x_j| < \sup\limits_{j\in\mathbb{N}} |x_j^N - x_j| < \epsilon/2, \quad \forall j \geq 1$ —(1)

Since $x_j^N \in c_0 \Rightarrow \lim\limits_{j\to\infty} x_j^N = 0.$

$j \to \infty$

Letting $j \to \infty$ in (1) $\Rightarrow$

$$|x_j| \le \epsilon/2 \quad \forall j \ge N.$$

$\Rightarrow x_j \to 0$ as $j \to \infty \Rightarrow x = (x_1 \dots x_j, \dots) \in C_0$.

Hence, $(C_0, \|\cdot\|_\infty)$ is a complete n.l.s.

Note: A complete n.l.s. $(X, \|\cdot\|)$ is called Banach space.

Ex. The space $(C[a,b], \|\cdot\|_\infty)$ is a complete n.l.s.

Let $f_n \in C[a,b]$ be a C.S. Then $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\|f_n - f_m\|_\infty < \epsilon/2$, $\forall n, m \ge N$

(*) $\longrightarrow$ $|f_n(t) - f_m(t)| < \epsilon/2$, $\forall n, m \ge N$.

For each fixed t, $f_n(t)$ is ~~a seq^n in R. (C.S.)~~ C.S. in $\mathbb{R}$. Therefore, $f_n(t) \to f(t) \in \mathbb{R}$. Since limit is unique f is well-defined.

Notice that N is independent of t. Letting $n \to \infty$ in (*) $\Rightarrow$ $|f(t) - f_m(t)| \le \epsilon/2 < \epsilon$, $\forall m \ge N$, $\forall t \in [a,b]$.

$\Rightarrow$ $|f(t) - f_N(t)| < \epsilon$, $\forall t \in [a,b]$.

Now, $|f(t) - f(s)| \le |f(t) - f_N(t)| + |f_N(t) - f_N(s)|$

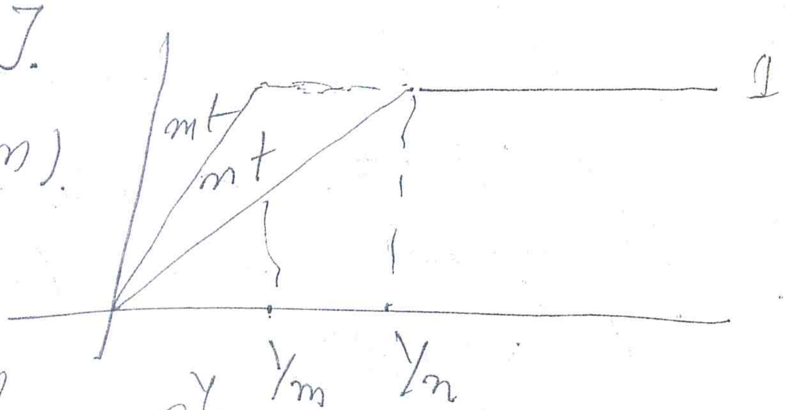
$+ |f_N(s) - f(s)| \; \angle \; 2\epsilon$

However, $(C[0,1], \|\cdot\|_1)$ is not complete.

Consider $f_n(t) = \begin{cases} nt & 0 \le t \le 1/n \\ 1 & 1/n \le t \le 1 \end{cases}$

Then $f_n \in C[0,1]$.

let $1/m < \frac{1}{n}$ $(n < m)$.



$$\|f_n - f_m\|_1 = \left(\int_0^{1/m} + \int_{1/m}^{1/n} + \int_{1/n}^{1}\right) |f_n(t) - f_m(t)| dt$$

$$= \int_0^{1/m} (mt - nt) + \int_{1/m}^{1/n} (1 - nt) + \int_{1/n}^{1} (1-1)$$

$$= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right) \to 0 \text{ as } n < m \to \infty.$$

$\Rightarrow f_n$ is C.C. in $(C[0,1], \|\cdot\|_1)$.

$f(t) = \lim f_n(t) = \begin{cases} 0 & t = 0 \\ 1 & 0 < t \le 1 \end{cases}$

which not cont on $[0,1]$.

note: $t = 0$, $f_n(0) = 0 \Rightarrow \lim f_n(0) = 0$.

$0 < t_0 < 1$, $t_0 > \frac{1}{n_0}$ for some n_0. $\Rightarrow n_0 t_0 > 1$.

$\Rightarrow t_0 > \frac{1}{n_0} > \frac{1}{n_0+1} \ldots \Rightarrow t_0 > \frac{1}{n}$, $n \ge n_0$.

$\Rightarrow f_n(t_0) = 1 \Rightarrow \lim f_n(t_0) = 1.$

Dense sets in (X,d):

A set $A \subset (X,d)$ is said to be dense in X if $\overline{A} = X$.

(i.e. $\forall x \in X$, $\exists x_n \in A$ st $x_n \to x$.
or $\forall x \in X$, $B_\epsilon(x) \cap A \neq \Phi$, $\forall \epsilon > 0$.)

~~Ex. $(\mathbb{R}, u) \supset \mathbb{Q}$ is~~

Ex $\overline{\mathbb{Q}} = \mathbb{R}$ with usual metric

$u(x,y) = |x-y|$.

let $x \in \mathbb{R}$, $x = [x] + \alpha$, $0 < \alpha < 1$.

But $\alpha = 0.x_1 x_2 \cdots\cdots$, $x_i \in \{0,1,2,\cdots,9\}$,

$$\Rightarrow x = x_0 + \frac{x_1}{10} + \frac{x_2}{10^2} + \cdots \infty$$

let $x_n = x_0 + \frac{x_1}{10} + \cdots + \frac{x_n}{10^n}$. Then $x_n \in \mathbb{Q}$. and

$$|x - x_n| = \frac{x_{n+1}}{10^{n+1}} + \cdots \to 0.$$

Thus, $x_n \in \mathbb{Q}$ & $x_n \to x \in \mathbb{R}$.

Ex. $1 \leq p < \infty$, then $\overline{C_{00}} = l^p$.

let $x \in l^p$, $x = (x_1, x_2, \cdots x_n, \cdots)$.

write $X^n = (x_1, x_2, \cdots x_n, 0, 0, \cdots)$. Then

$X^n \in c_{00}$, $\forall\ n \geq 1$. (97)

now,

$$\|x - X^n\|_p = \left(\sum_{k=n}^{\infty} |x_{k+1}|^p\right)^{1/p} \to 0 \text{ as } n \to \infty.$$

$\Rightarrow X^n \to x$.

Ex. $\overline{c_{00}} = c_0$. Let $x \in c_0$. Then

$x = (x_1, x_2, \cdots x_n, \cdots)$ &

$\lim_{n\to\infty} x_n = 0$. For $\epsilon > 0$, $\exists\ N \in \mathbb{N}$

st $|x_n| < \epsilon/2$, $\forall\ n \geq N$. — (1)

write $X^n = (x_1, x_2, \cdots x_n, 0, \cdots)$, $n \geq N$.

Then $X^n \in c_{00}$. and

$$\|x - X^n\|_\infty = \sup_{n \geq N} |x_{n+1}| \leq \epsilon/2 \quad (\text{by } (1)), \quad \forall\ n \geq N.$$

$\Rightarrow X^n \to x$.

Remark: $\overline{c_{00}} = c_0 \subsetneq l^\infty$. That is, c_{00} is not dense in l^∞.

Ex. $(\mathbb{R}, d)$ with $d(x,y) = |\tan^{-1}x - \tan^{-1}y|$ is incomplete. Hint: $x_n = \tan \frac{\pi}{2}\left(\frac{n}{n+1}\right)$ is C.S. but not converging to a pt of $\mathbb{R}$.

Ex. Every discrete metric space is complete.

let $X \neq \phi$, and $d_0(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$

Suppose $(x_n) \subset X$ be C-S. Then for $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ st

$$d_0(x_n, x_m) < \epsilon, \ \forall\ n, m \geq N.$$

Now, $d_0(x_n, x_m) = \begin{cases} 0 & \text{if } 0 < \epsilon \leq 1. \\ 0 \text{ or } 1 & \text{if } \epsilon > 1. \end{cases}$

But if $d_0(x_n, x_m) = 1$ for inf'ly many $n, m > N$ (for some $\epsilon > 1$), then

$$\lim_{n,m \to \infty} (x_n, x_m) = 1 \neq 0 \text{ (why?)}.$$

Thus, for $\forall \epsilon > 0$, $\exists$ $N' \in \mathbb{N}$ st

$$d(x_n, x_m) = 0, \ \forall\ n, m \geq N'.$$

i.e. $(x_n) = (x_1, x_2, \ldots, x_{N'}, x, x, \ldots) \rightarrow x.$

(i.e every C.S. in (X, d_0) is almost constant.)

Continuous maps on (X, d):

A function $f: (X, d) \longrightarrow (\mathbb{R}, \mathcal{U})$ is said to be cont at $x_0 \in X$ if $\forall \epsilon > 0$

$\exists\ \delta > 0$ st $d(x_0, y) < \delta \Rightarrow |f(x_0) - f(y)| < \epsilon$

$$f(B_\delta(x_0)) \subseteq (f(x_0) - \epsilon,\ f(x_0) + \epsilon).$$

<u>Theorem</u>: If $f : (X, d) \longrightarrow (\mathbb{R}, \varphi)$ & ($\mathbb{R}$ just).

Then FAE.

(i) f is cont on X (with ϵ–δ, def^n).

(ii) $\forall\ x_n \in X$ st $x_n \longrightarrow x \Rightarrow f(x_n) \to f(x)$.

(iii) $f^{-1}(O)$ is open in $(\cancel{\mathbb{R}}, d)$, $\forall$ open set $O \subseteq \mathbb{R}$.

(iv) $f^{-1}(F)$ is closed in (X, d), $\forall$ closed set $F \subseteq \mathbb{R}$.

(Proof is similar as to $f : \mathbb{R} \longrightarrow \mathbb{R}$ when $u \to d$. $u(x,y) \to d(x,y)$).

Ex. for $x, y, z \in (X, d)$, we get

$$|d(x,y) - d(x,z)| \leq d(y,z).$$

(by Δ-inequality)

Thus, for $f(y) = d(x_0, y)$.

$$|f(y) - f(z)| \leq d(y,z) \to 0 \text{ as } y \to z.$$

Hence, f is cont on (X, d) to $(\mathbb{R}, u)$.

Def^n: A function $f: A \subset (X,d) \longrightarrow \mathbb{R}$ is said to be uniformly cont on A if $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ st

100

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Note that δ is free of choice of location of pts $x, y \in A$ (rather only depends upon how far ~~they are~~ x & y are separated.)

Ex. For $x_0 \in X$, $f(x) = d(x, x_0)$ is uniformly cont on X,

Ex. For $x \in X$, & $A \subset X$, we define

$$d(x, A) = \inf_{a \in A} d(x, a),$$ called distance of A from x, is unif. cont on X.

Because $d(x,a) \leq d(x,y) + d(y,a)$

$$\Rightarrow \inf_{a \in A} d(x,a) \leq d(x,y) + \inf_{a \in A} d(y,a)$$

$$\Rightarrow |d(x,A) - d(y,A)| \leq d(x,y).$$

Thus, $f(x) = d(x,A)$ is unif. cont. on X.

<u>Result</u>: $f: (X,d) \longrightarrow \mathbb{R}$ is unif cont on $A \subseteq X$ iff $\forall\ x_n, y_n \in A$ st $d(x_n, y_n) \to$

$\Rightarrow |f(x_n) - f(y_n)| \to 0$ as $n \to \infty$.

Pf: Suppose f is unif cont on $A \subseteq X$, and $x_n, y_n \in A$ st $d(x_n, y_n) \to 0$. Then for $\epsilon > 0$, $\exists\ \delta > 0$ st

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon. \quad -(1)$$

Since $d(x_n, y_n) \to 0$. For $\delta > 0$, $\exists\ N \in \mathbb{N}$ st $d(x_n, y_n) < \delta$, $\forall\ n \geq N$. By (1),

$$|f(x_n) - f(y_n)| < \epsilon, \quad \forall\ n \geq N.$$

$$\Rightarrow |f(x_n) - f(y_n)| \to 0 \text{ as } n \to \infty.$$

Conversely, suppose for some $\epsilon_0 > 0$, and $\forall\ \delta > 0$, $\exists\ x, y \in A$ st

$$d(x,y) < \delta \text{ but } |f(x) - f(y)| \geq \epsilon_0.$$

Consider $\delta = \frac{1}{n} > 0$, $n \in \mathbb{N}$. Then $\exists\ x_n, y_n \in A$ st $d(x_n, y_n) < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$.

ie $d(x_n, y_n) \to 0$ but $\lim_{n\to\infty} |f(x_n) - f(y_n)| \geq \epsilon_0 > 0$.

Thus, $\nexists$ such $\epsilon_0 > 0$. That is, f is unif cont on A.

Ex. $f : (0,1] \longrightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$, is cont but not unif cont, $x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$

$|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| = 1 \not\to 0$.

Ex. (i) $f : \mathbb{N} \longrightarrow \mathbb{R}$ is unif cont.

(ii) $f : [0,1] \longrightarrow \mathbb{R}$, $f(x) = x^{\alpha}$ is unif cont if $0 < \alpha \leq 1$, and not unif cont for $\alpha > 1$.

(iii) $f : (0, \infty) \longrightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$ is not unif cont.

Result: If $f : [a,b] \longrightarrow \mathbb{R}$ is cont, then f is unif cont.

If not, let $x_n, y_n \in [a,b]$ st $|x_n - y_n| \to 0$, but $|f(x_n) - f(y_n)| \geq \epsilon_0$, $\forall n \geq 1$. —(1)

By B-W theorem, $\exists$ $x_{n_k} \to x$ & $y_{n_k} \to y$.

$$\Rightarrow |x_{n_k} - y_{n_k}| \to |x - y|$$

But $|x_n - y_n| \to 0 \Rightarrow |x - y| = 0$ i.e. $x = y$.

Now, from (1), $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0$

Since f is cont, ~~$\Rightarrow$ and~~ $|f(x) - f(x)| \geq \epsilon_0$ X.

Result: If $f : \mathbb{R} \longrightarrow \mathbb{R}$ st f is unif ~~$f : (a,b]$~~ cont on $(a,b]$ & $[b,c)$.

Then f is unif cont on (a,c).

$f : (a,b] \cup (b,c) \longrightarrow \mathbb{R}$.

we need to verify that $\forall \epsilon > 0$,

$\exists \delta > 0$ st

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

(i) if $x, y \in (a,b]$ or $[b,c)$ ok.

(ii) if $x \in (a,b]$, $y \in [b,c)$ st $|x-y| < \delta$.

Then $|x-b| < \delta \Rightarrow |f(x) - f(b)| < \epsilon$

$|b-y| < \delta \Rightarrow |f(y) - f(b)| < \epsilon$

$\Rightarrow |f(x) - f(y)| < 2\epsilon$.

Ex. If $f : (X,d) \longrightarrow \mathbb{R}$ is unif cont,

then f sends C.C in X to C.C in $\mathbb{R}$.

Let (x_n) be a C.C in (X,d). ~~Then~~

Since f is unif cont, for $\epsilon > 0$, $\exists \delta > 0$

s/t $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

for $\delta > 0$, $\exists N \in \mathbb{N}$ st $d(x_n, x_m) < \delta$, $\forall n, m \geq N$

$\Rightarrow |f(x_n) - f(x_m)| < \epsilon$, $\forall n, m \geq N$

ie $f(x_n)$ is C.C in $\mathbb{R}$.

Contraction map on (X,d):

A map $f:(X,d)\longrightarrow(X,d)$ is said to be contraction, if $\exists$ $k\in(0,1)$ s.t.

$$d(f(x),f(y))\leq k\,d(x,y),\quad \forall\, x,y\in X.$$

Theorem: Let (X,d) be a complete metric space and $\phi:(X,d)\longrightarrow(X,d)$ be contraction, then $\exists!\, x_0\in X$ s.t. $\phi(x_0)=x_0$.

Pf: Let $x_0\in X$ & $\phi(x_n)=x_{n+1}$, $n\geq 0$

~~Then $\phi(x_n, x_{n+1})\leq k$~~

$$d(\phi(x_n),\phi(x_{n+1}))\leq k\,d(\phi(x_{n-1}, x_n))$$
$$\leq k^n d(x_0,x_1).$$

If $m>n$,

$$d(\phi(x_n),\phi(x_m))\leq \frac{k^n}{1-k}d(x_0,x_1)\longrightarrow 0 \text{ as } n\to\infty.$$

$\Rightarrow \{x_{n+1}\}$ is a C.S. in (X,d), thus has limit say $x_{n+1}\to x$. But $\lim \phi(x_n)=x$

$\Rightarrow \phi(x)=x$. If $\exists$ $y\in X$ s.t. $\phi(y)=y$. Then

$$d(x,y)\leq k\,d(x,y),\quad k\in(0,1)\ \times.$$

Uniform convergence of seq^{ns}:

Consider the seq^n $f_n : A \subset \mathbb{R} \longrightarrow \mathbb{R}$

we say f_n converges to $f : A \subset \mathbb{R} \to \mathbb{R}$ point wise if ~~for the...~~ for any $t_0 \in A$, and $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ st

$$|f_n(t_0) - f(t_0)| < \epsilon, \quad \forall n \geq N.$$

notice that $N_0 = N(\epsilon, t_0)$.

Ex. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(t) = e^{-nt^2}$, $n \in \mathbb{N}$.

Then $f(t) = \begin{cases} 1 & t = 0 \\ 0 & |t| > 0 \end{cases}$

$$|f_n(0) - f(0)| = |1 - 1| = 0 < \epsilon, \quad \forall n \geq 1.$$

now, $|t_0| > 0$, $t_0^2 > 0$. Then for

$$|f_n(t_0) - 0| < \epsilon \Rightarrow e^{-nt_0^2} < \epsilon$$

$$\Rightarrow n > \frac{\log 1/\epsilon}{t_0^2}, \text{ let } N_0 = \left[\frac{\log 1/\epsilon}{t_0^2}\right] + 1$$

Then $N_0 = N_0(\epsilon, t_0)$. & N_0 is larger when $|t_0|$ is close to "0". Thus, N_0 cannot be free of t_0!

However, if it happens that N_0 is free of choice of $t_0 \in A$. Then, we say f_n converges to f uniformly. (106)

Note: $f_n \to f$ unif, if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ ~~A~~

$$|f_n(t) - f(t)| < \epsilon, \quad \forall n \geq N, \forall t \in A.$$

~~If $A = [a,b]$ (or a compact set in $\mathbb{R}$~~).

Then $$\sup_{t \in A} |f_n(t) - f(t)| \leq \epsilon, \quad \forall n \geq N.$$

$$\text{i.e.} \quad \|f_n - f\|_\infty \leq \epsilon, \quad \forall n \geq N.$$

$$\text{i.e.} \quad \|f_n - f\|_\infty \to 0 \text{ as } n \to \infty.$$

If $f_n(t) = e^{-nt^2}$, $t \in \mathbb{R}$, $n \in \mathbb{N}$

$$\sup_{t \in \mathbb{R}} |f_n(t) - f(t)| = 1 \not\to 0.$$

Hence $f_n \to f$ pt wise but <u>not</u> unif.

Ex. If $f_n, f : A \subset \mathbb{R} \to \mathbb{R}$ be such that $f_n \to f$ unif. Then for ~~$f_n(t)$~~ $|f_n(t)| \leq M_n$ implies f is bounded.

For $\epsilon = 1$, $|f(t)| \leq |f(t) - f_N(t)| + |f_N(t)|$

$$\leq 1 + M_N.$$

Ex. If $f_n \to f$ unif & f_n are cont / unif cont, then f is cont / unif cont.

Result: Let $f_n, f \in R[a,b]$ be s.t. $f_n \to f$ unif on $[a,b]$. Then

$$\int f_n \to \int f \quad \text{or} \left(\lim \int f_n = \int \lim f_n \right)$$

Pf: $$\left| \int_a^b (f_n - f) \right| \le \int_a^b |f_n - f| \le \|f_n - f\|_\infty (b-a).$$

Since $f_n \to f$ unif $\Rightarrow \|f_n - f\|_\infty < \epsilon$ $\forall n \geq N$

ie $\left| \int_a^b f_n - \int_a^b f \right| < \epsilon (b-a)$, $\forall n \geq N$

ie $\int_a^b f_n \to \int_a^b f$.

Cor: If $f_n \in R[a,b]$ st $S_n = f_1 + \cdots + f_n \to S$ unif, then $\int \sum f_n = \sum \int f_n$.

(Obvious from previous result).

~~Result: If~~

Result: If $f_n \in C^1[a,b]$ st $f_n' \to g$ unif & $\{f_n(x_0)\}$ is conv. for some x_0, Then $f_n \to f$ unif. with $f' = g$.

Remark: Conv. of $\{f_n(x_0)\}$ is necessary in the above result.

Ex. $f_n(\cancel{t}) = \sqrt{\cancel{t}+n}$, $t \in [0, \infty)$. Then f_n does not converge at any pt $t \in [0,\infty)$.

However, $f_n'(t) = -\dfrac{1}{2\sqrt{t+n}} \xrightarrow{\text{unif.}} 0$.

Since $\|f_n' - 0\|_\infty = \sup\limits_{t \geq 0} \left| \dfrac{1}{2\sqrt{t+n}} \right| = \dfrac{1}{2\sqrt{n}} \to 0$.

Ex. Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$. ~~Show that~~ Check the unif conv. of f_n to some f.

(i) $f_n(t) = \dfrac{\sin nt}{\sqrt{n}}$

(ii) $f_n(t) = n^2 \cancel{t} (1 - \cancel{t^2})^n$

(iii) $f_n(t) = t e^{-nt}$

Verify term by term integration / differentiate for each.