Notations:

(i) $L_n(\mathbb{R})$ = Space of all linear maps from $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

(ii) $GL_n(\mathbb{R}) = \{A \in L_n(\mathbb{R}) : AA^{-1} = I\}$
= Set of all invertible matrices

Result: Let $A \in GL_n(\mathbb{R})$ and $B \in L_n(\mathbb{R})$ be such that $\|B-A\| < \frac{1}{\|A^{-1}\|}$. ~~then $B \in GL_n(\mathbb{R})$~~

Then (i) $B \in GL_n(\mathbb{R})$

(i.e $GL_n(\mathbb{R})$ is open in $L_n(\mathbb{R})$).

(ii) $A \longmapsto A^{-1}$ is cont on $GL_n(\mathbb{R})$.

proof: Let $\alpha = \frac{1}{\|A^{-1}\|}$, $\beta = \|B-A\|$. Then $\beta < \alpha$. For $x \in \mathbb{R}^n$, write

$$\alpha\|x\| = \alpha\|A^{-1}Ax\| \leq \alpha\|A^{-1}\|\,\|Ax\|$$

$$\text{i.e. } \alpha\|x\| \leq \|Ax\| = \|(A-B)x + Bx\|$$
$$\leq \|A-B\|\|x\| + \|Bx\|$$

$$(\alpha-\beta)\|x\| \leq \|Bx\| \quad \text{——(1)}$$

(i) If $Bx = 0$, then $(\alpha-\beta)\|x\| = 0 \Rightarrow x = 0$.

Since B is a 1-1 linear map from $\mathbb{R}^n \to \mathbb{R}^n$, B is onto.

(ii) Put $x = B^{-1}y$ in (1), Then

$$\frac{\|B^{-1}y\|}{\|y\|} \leq \frac{1}{\alpha-\beta} \; ; \; y \neq 0.$$

$$\sup_{y\neq 0} \frac{\|B^{-1}y\|}{\|y\|} \leq \frac{1}{\alpha-\beta} \Rightarrow \|B^{-1}\| \leq \frac{1}{\alpha-\beta}.$$

Now,

$$\|B^{-1}-A^{-1}\| = \|B^{-1}(A-B)A^{-1}\|$$
$$\leq \frac{\|A-B\|}{\alpha(\alpha-\beta)} \to 0 \text{ as } A \to B.$$

Hence, the map $A \mapsto A^{-1}$ is cont.

note that $A \to A^{-1}$ is 1-1 map, because

$$A^{-1} = B^{-1} \Rightarrow A = B.$$

Contraction mapping:

Let $D \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$.
A map $f : D \to D$ is said to be contraction if $\exists$ $K \in (0,1)$, s.t.

$$\|\varphi(x) - \varphi(y)\| \leq K \|x-y\|.$$

Ex. is uniformly continuous.

Ex. Let $\varphi : (0,\infty) \to (0,\infty)$ by

$$\varphi(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \; a > 0.$$

Then φ is not a contraction, though $\varphi(\sqrt{a}) = \sqrt{a}$. (fixed pt)

For $x \neq y$, $|\varphi(x) - \varphi(y)| = \frac{1}{2}\left|1 - \frac{a}{xy}\right| |x-y|$ is not bdd near zero.

Hence can not be less than 1.

Ex $\varphi : (0, 2\pi) \to (0, 2\pi)$. $\varphi(x) = \sin\frac{x}{2}$

$|\varphi(x) - \varphi(y)| \le \frac{1}{2}|x-y|$ (By MVT)

Thus, φ is a contraction mapping, but φ has no fixed pt in $(0, 2\pi)$.

<u>Lemma:</u> Let B be a closed subset of $\mathbb{R}^n$, and $\varphi : B \to B$ is a contraction mapping. Then φ has a unique fixed point in B.

<u>pf:</u> For $x_0 \in B$, define a seq^n

$$x_{n+1} = \varphi(x_n), \quad n \geq 0.$$

Then $\|x_{n+1} - x_n\| \le K^n \|x_1 - x_0\|$ ($\because$ Ex)

If $m > n$, then

$$\|x_m - x_n\| \le \frac{K^n}{1-K}\|x_1 - x_0\| \to 0 \text{ as } n \to \infty.$$

Then $\{x_n\}$ is a Cauchy seq^n in B and B is closed, implies $x_n \to x \in B$.

Then $\varphi(x) = \lim \varphi(x_n) = \lim x_{n+1} = x$

$\Rightarrow \varphi$ has a fixed pt

If $y = \varphi(y)$. then

$$\|x - y\| = \|\varphi(x) - \varphi(y)\| < K\|x - y\|.$$

which is not true since $0 < K < 1$.

$\Rightarrow$ unique fixed pt

Remark: If $\Omega \subset \mathbb{R}^n$ is open, then any contraction mapping $f: \Omega \to \Omega$ can have at most one fixed pt.

Ex. Let $f: \mathbb{R} \xrightarrow[\text{onto}]{1-1} \mathbb{R}$ and f is continuously diff at $x_0 \in \mathbb{R}$ st $f'(x_0) \neq 0$. Then f^{-1} is diff at $y_0 = f(x_0)$ & $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$

$$\varepsilon(K) = \frac{f^{-1}(y_0+K) - f^{-1}(y_0) - \frac{K}{f'(x_0)}}{|K|}$$

Let $h = f^{-1}(y_0+K) - f^{-1}(y_0)$. $y_0+K = f(x_0+h)$

& $K = f(x_0+h) - f(x_0) \Rightarrow h f'(x_0+\theta h)$ —(1)

Since $f'(x_0) \neq 0$, $\exists\, \delta > 0$ st $f'(x) \neq 0$, $\forall x \in [x_0-\delta, x_0+\delta]$.

ie $|f'(x)| > m$, $\forall x \in [x_0-\delta, x_0+\delta]$.

choose h small st $x_0+\theta h \in [x_0-\delta, x_0+\delta]$.

$|K| > |h| m$. Thus $K \to 0 \Rightarrow h \to 0$.

$$|\varepsilon(K)| = \frac{\left| h - \frac{f(x_0+h) - f(x_0)}{f'(x_0)} \right|}{|f(x_0+h) - f(x_0)|} = \frac{|f'(x_0) - f'(x_0+\theta h)|}{|f'(x_0+\theta h)|\,|f'(x_0)|}$$

$\because f'$ is cont at x_0, $\longrightarrow \frac{0}{|f'(x_0)|} = 0$.

note: If f^{-1} is diff then $f^{-1} \circ f(x) = x$, $(f^{-1})'(f(x_0)) f'(x_0) = 1$

Inverse mapping theorem:

Let Ω be an open set in $\mathbb{R}^n$. Suppose $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1-map s.t. $\det f'(x_0) \neq 0$. Then

(i) $\exists$ open sets U & $V \subset \mathbb{R}^n$ s.t.
$f : U \longrightarrow V \ (= f(U))$ is 1-1, onto.

(ii) f^{-1} is a C^1-map on V &
$$(f^{-1})'(f(x_0)) = (f'(x_0))^{-1}.$$

Proof: Let $A = f'(x_0)$. For $y \in \mathbb{R}^n$, define
$$\varphi : \Omega \longrightarrow \mathbb{R}^n \text{ by}$$
$$\varphi(x) = x + A^{-1}(y - f(x)). \quad \text{---} (1)$$
Then $\varphi(x) = x$ iff $y = f(x)$.
(i.e x is the fixed pt of φ iff $y = f(x)$)

Since f' is cont at x_0, for $\epsilon = \frac{1}{2\|A^{-1}\|} > 0$ $\exists$ $\delta > 0$ s.t.
$$\|x - x_0\| < \delta \Rightarrow \|f'(x) - f'(x_0)\| < \frac{1}{2\|A^{-1}\|}.$$
Let $U = B_\delta(x_0) = \{x \in \Omega : \|x - x_0\| < \delta\}$.
and $V = f(U)$.

(i) Claim f is 1-1 on V.

Now, $\phi'(x) = I - A^{-1} f'(x)$

$= A^{-1}(A - f'(x))$

$\therefore \|\phi'(x)\| \leq \|A^{-1}\| \|A - f'(x)\| < \frac{1}{2}$

If $x_1, x_2 \in V$, by MVT for ϕ,

$\|\phi(x_1) - \phi(x_2)\| \leq \|\phi'(x_1 + \lambda(x_2 - x_1))\| \|x_1 - x_2\|$

ie $\|\phi(x_1) - \phi(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\|$.

$\Rightarrow \phi$ is a contraction on V. Hence ϕ can have only one fixed pt, hence $y = f(x)$ for atmost one $x \in V$.

$\Rightarrow f$ is 1-1 on V.

(ii) <u>V' is open</u>: let $y^* \in V'$. Then $y^* = f(x^*)$ for some $x^* \in V$.

Then $\exists\, r > 0$ s.t $B_r(x^*) = \{x \in V : \|x - x^*\| < r\} \subset V$.

Now, it is enough to prove that, whenever

$$\|y - y^*\| < \frac{r}{2\|A^{-1}\|} \Rightarrow y \in V'. \quad \text{——(2)}$$

Suppose $\|y - y^*\| < \frac{r}{2\|A^{-1}\|}$

(PO)

Then
$$\|\phi(x^*) - x^*\| = \|A^{-1}(y - y^*)\|$$
$$\leq \|A^{-1}\| \, \|y - y^*\| < \frac{r}{2}.$$

If $x \in \overline{B_r}(x^*) = \{x \in \Omega : \|x - x^*\| \leq r\}$, then
$$\|\phi(x) - x^*\| \leq \|\phi(x) - \phi(x^*)\| + \|\phi(x^*) - x^*\|$$
$$< \frac{1}{2}\|x - x^*\| + \frac{r}{2} \leq r.$$

$$\Rightarrow x \in \overline{B_r}(x^*) \Rightarrow \phi(x) \in \overline{B_r}(x^*).$$

$\phi : \overline{B_r}(x^*) \longrightarrow \overline{B_r}(x^*)$ is a Contraction mapping. Then ϕ has a fixed pt $x \in \overline{B_r}(x^*)$ s.t. $\phi(x) = x$ iff $y = f(x)$.

Now $y = f(x) \subset f(\overline{B_r}(x^*)) \subset f(U) = V$.

Thus, V is open and hence
$$f : U \xrightarrow[\text{onto}]{1-1} V \,(= f(U)) \text{ - open.}$$

(iii) $f^{-1} : V \longrightarrow U$ is diff. at $f(x_0)$.

Let $y \in V$, then $y + k \in V$ ($\because V$ is open) for small $\|k\|$.

Let $h = f^{-1}(y+k) - f^{-1}(y)$. Then
$$k = f(x+h) - f(x) \quad (\because f^{-1}(y) = x)$$

Now, $\phi(x+h) - \phi(x) = h + A^{-1}(f(x) - f(x+h))$
$$= h - A^{-1}k.$$

$\Rightarrow \|h - A^{-1}k\| \leq \frac{1}{2}\|h\|.$

$\Rightarrow \|h\| \leq \|h - A^{-1}k\| + \|A^{-1}k\|$

$\leq \frac{1}{2}\|h\| + \|A^{-1}k\|$

ie $\frac{1}{2}\|h\| \leq \|A^{-1}k\|$ —— (3)

$\leq \|A^{-1}\|\|k\|$

now,

$$\eta(k) = \frac{f^{-1}(y_0+k) - f^{-1}(y_0) - (f'(x_0))^{-1}k}{\|k\|}$$

$$= \frac{(f'(x_0))^{-1}\{f'(x_0)h - (f(x_0+h) - f(x_0))\}}{\|k\|}$$

$$\therefore \|\eta(k)\| \leq \frac{\|f'(x_0)^{-1}\| \, \|f(x_0+h) - f(x_0) - f'(x_0)h\|}{\|h\| / 2\|A^{-1}\|}$$

$\to 0$ as $h \to 0$ ($\because k \to 0 \Rightarrow h \to 0$).

$\Rightarrow (f^{-1})'(f(x_0)) = (f'(x_0))^{-1}$

(iv) f^{-1} is continuously diff. ie,

$(f^{-1})'$ is cont. need to prove.

Since $(f^{-1})'(y_0) = (f'(x_0))^{-1}$.

Since $A \longmapsto A^{-1}$ is cont on $GL_n(\mathbb{R})$

$(f^{-1})'$ is cont.

Ex. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$f(x,y) = (x - e^{-y}, y - e^{x})$. Then

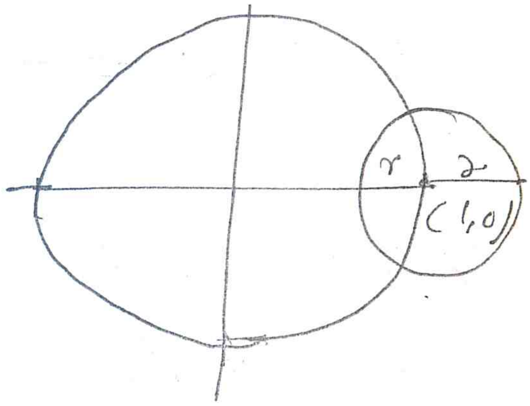
$f'(0,0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\det f'(0,0) = 2 \neq 0$.

Hence f is 1-1, in a nhd of $(0,0)$ &

$(f^{-1})'(f(0,0)) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$.

Implicit function theorem:

Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by



$f(x,y) = x^2 + y^2 - 1$

Then $f'(x,y) = (2x \quad 2y)$.

$\left.\frac{\partial f}{\partial x}\right|_{(1,0)} = 2$, $\left.\frac{\partial f}{\partial y}\right|_{(1,0)} = 0$.

Then one can draw a ball around $(1,0)$ s.t. of radius $r < 1$ st $f(\phi(y), y) = 0$, why

$x = \phi(y)$, $|y| < r < 1$. $\phi(y) = \sqrt{1-y^2}$

However, we cannot draw ball of any radius around $(1,0)$ st $f(x, \psi(x)) = 0$. why

$y = \psi(x)$, for $|x| < r$, r even very small.

Because, for any $r > 0$, we cannot write

$\psi(x) = \sqrt{1-x^2}$ as $x > 1$

will be included in any ball around $(1,0)$.

However, at any pt on the circle, other than $(\pm 1, 0)$ & $(0, \pm 1)$, we can solve x & y simultaneously in a small nhd. of the pt.

65

Now, consider a linear map

$$A : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n.$$

Then $(h,k) \in \mathbb{R}^n \times \mathbb{R}^m$, $(h,k) = (h,0) + (0,k)$.

$$A(h,k) = A(h,0) + A(0,k)$$

$$= A_x h + A_y k. \quad \text{(say)}$$

<u>~~Lemm~~ Lemma:</u> If A_x is invertible ($A_x \in L_n(\mathbb{R})$) then for each $k \in \mathbb{R}^m$, $\exists$ unique $h \in \mathbb{R}^n$, s.t. $h = -A_x^{-1} A_y k$.

<u>proof:</u> $A(h,k) = 0$ iff $A_x h + A_y k = 0$. Since A_x is invertible, $h = -A_x^{-1} A_y k$.

Now, let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set.

and $f : \Omega \subset \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be diff.

$f = (f_1, \cdots f_n)$;

$$f_i : \Omega \subset \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$$

$$f_i'(x,y) = \left(\frac{\partial f_i}{\partial x_1}(x,y) \cdots \frac{\partial f_i}{\partial x_n}(x,y) \; \frac{\partial f_i}{\partial y_1}(x,y) \cdots \frac{\partial f_i}{\partial y_m}(x,y) \right)$$

$$f' = \left[\begin{array}{ccc|ccc} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & & & \vdots & & \\ \underbrace{\frac{\partial f_n}{\partial x_1} \quad \cdots \quad \frac{\partial f_n}{\partial x_n}}_{n \times n} & & & \underbrace{\frac{\partial f_n}{\partial y_1} \quad \cdots \quad \frac{\partial f_n}{\partial y_m}}_{n \times m} & & \end{array}\right]$$

$$= \left[\left(\frac{\partial f_i}{\partial x_j}\right)_x \ \left(\frac{\partial f_i}{\partial y_k}\right)_y \right]_{n \times (n+m)}.$$

$$= (A_x \ \ A_y).$$

Then $A_x : \mathbb{R}^n \xrightarrow{\text{linear}} \mathbb{R}^n$ & $A_y : \mathbb{R}^m \xrightarrow{\text{linear}} \mathbb{R}^n$,

when $A_x = \left(\frac{\partial f_i}{\partial x_j}\right)_x$, $A_y = \left(\frac{\partial f_i}{\partial y_k}\right)_y$.

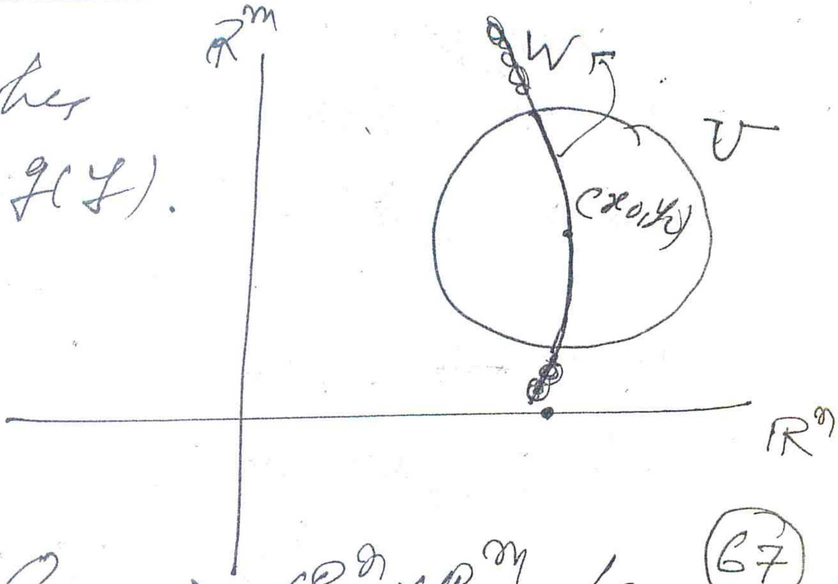
<u>Theorem (Implicit function theorem)</u>:

Let Ω be an open subset in $\mathbb{R}^n \times \mathbb{R}^m$. If $f : \Omega \subset \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a C^1-map, with $f(x_0, y_0) = 0$ & $\det [f'(x_0, y_0)]_x \neq 0$. for some $(x_0, y_0) \in \Omega$. Then

(i) $\exists$ open sets $V \subset \mathbb{R}^n \times \mathbb{R}^m$ & $W \subset \mathbb{R}^m$ s.t. $\forall y \in W$, $\exists !\ x \in \mathbb{R}^n$ ~~st~~ with $(x, y) \in V$ and $f(x, y) = 0$.

(ii) if $x = g(y)$, then $g : W \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is C^1-map, $g(y_0) = x_0$, $f(g(y), y) = 0$, $\forall y \in W$. and $g'(y_0) = -A_x^{-1} A_y$, $A = f'$.

i.e. f will vanish on a curve $x = g(y)$.



Proof: Let $F : \Omega \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$ by

67

$$F(x,y) = (f(x,y), y).$$ Then F is a C^1-map. and

$$F'(x_0,y_0) = \begin{bmatrix} \{f'(x_0,y_0)\}_x & \{f'(x_0,y_0)\}_y \\ 0 & I \end{bmatrix}$$

$\det F'(x_0,y_0) \neq 0$. Therefore, by IMT, $\exists$ open set U & $V \subset \mathbb{R}^n \times \mathbb{R}^m$ such that

$$F : U \xrightarrow[\text{onto}]{1-1} V \quad C^1\text{-map}.$$

Let $W = \{ y \in \mathbb{R}^m : (0,y) \in V \}$. Then W is open, because V is open.

Since F is onto, for $y \in W$,

$$(0,y) = F(x,y) \Rightarrow (x,y) \in U.$$

$$\Rightarrow f(x,y) = 0, \ \forall y \in W.$$

Suppose, for this y, $\exists$ $(x',y) \in V$

such that $f(x', y) = 0$. Then (G8)

$F(x', y) = (f(x', y), y) = (f(x, y), y) = F(x, y)$.

Since F is 1-1 on V, $\Rightarrow x' = x$.

(ii) Define $x = g(y)$, for $y \in W$. Then

$(g(y), y) \in V$ and $f(g(y), y) = 0$ — (*)

Then $F(g(y), y) = (0, y)$, $\forall y \in W$

ie $F^{-1}(0, y) = (g(y), y)$

By ~~by~~ IMT, F^{-1} is a C^1-map,

hence g is a C^1-map.

To compute $g'(y_0)$, consider

$f(g(y), y) = 0$, $y \in W$

differentiating w.r.t y and using chain rule, we get

$$f'(g(y_0), y_0)\begin{pmatrix} g'(y_0) \\ I \end{pmatrix} = 0$$

$$f'(x_0, y_0)\begin{pmatrix} g'(y_0) \\ I \end{pmatrix} = 0$$

$$(A_x \; A_y)\begin{pmatrix} g'(y_0) \\ I \end{pmatrix} = 0$$

$$A_x g'(y_0) + A_y = 0$$

$$\Rightarrow g'(y_0) = A_x^{-1} A_y.$$

Ex. Show that $x^2 + ye^x - \sin(xy) = 0$ can be solved for y in a nbd of $(0,0)$ but cannot be solved for x in <u>any</u> nbd of $(0,0)$.

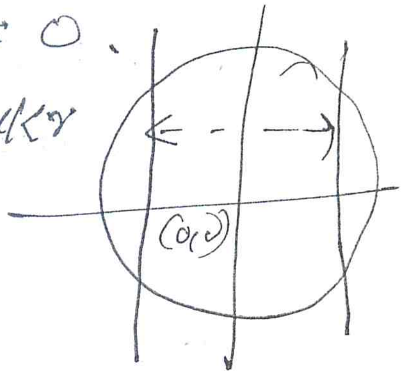
$$F(x,y) = x^2 + ye^x - \sin xy \quad \text{---(1)}$$

(i) $F(0,0) = 0$, $\left.\frac{\partial F}{\partial y}\right|_{(0,0)} = 1 \neq 0$,

By implicit fn. ~~theorem~~ theorem $\exists$ a ball around $(0,0)$ and an interval for x s.t

$$F(x, g(x)) = 0.$$

or $y = g(x)$, for $|x|<r$



(ii) $\left.\frac{\partial F}{\partial x}\right|_{(0,0)} = 0.$

Hence implicit function theorem <u>cannot</u> be applied.

On contrary, suppose $x = \phi(y)$, then

$$0 = \phi(0). \;\&$$

$$(\phi(y))^2 + y e^{\phi(y)} - \sin(\phi(y) y) = 0$$

for $|y|<r$, for some $r>0$

Then $2\phi(0)\phi'(0) + 1\cdot e^{\phi(0)} + 0\cdot e^{\phi(0)}\phi'(0)$

$$- \cos(\phi(0)\,0)\,(\phi'(0)\,0 + \phi(0)\cdot 1) = 0$$

$$\Rightarrow \quad 1 = 0 \quad \times$$

Ex. $f: \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f(x,y,z) = (xe^y + ye^z,\ xe^z + ze^y)$.

Then f is a C^1-map.

$$f'(x,y,z) = \begin{pmatrix} e^y & xe^y + e^z & ye^z \\ e^z & ze^y & xe^z + e^y \end{pmatrix}$$

$f(-1,1,1) = (0,0)$.

Let $f = (f_1, f_2)$. Then

$$\begin{pmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix}(-1,1,1) = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}$$

By implicit function theorem, ∃ open ball U in $\mathbb{R}^3$ and open ball V in $\mathbb{R}^2$ s.t. ~~$x = \phi(x)$~~

$$(y,z) = (\phi(x), \psi(x)), \quad |x| < \gamma,$$

for some $\gamma > 0$.

Ex. Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a C^1-map s.t. $f(0,0) = 0$, $f_x(0,0) = 1$. Let $F(x,y) = (f(x,y), y)$. Show that F is inj. in some nbd of $(0,0)$. Does F is inj in any nbd of $(0,0)$?

Remark: Condition in Implicit/Inverse function theorems on derivative are sufficients.

Ex. $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x,y) = x^2 - y^3$

$f(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0$, but $y = x^{2/3}$ is a solution of $f(x,y) = 0$ near $(0,0)$.

Ex. Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x,y) = (x^3, y^3)$.

then $\det f'(0,0) = 0$ but f is 1-1, onto.