DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA224: Real Analysis — End Semester Exam
Instructor: Rajesh Srivastava — May 3, 2018
Time duration: 03 hours — Maximum Marks: 45

N.B. Answer without proper justification will attract zero mark.

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1. (a) Does there exist am unbounded open set $A \subset \mathbb{R}$ such that $m(A) < \infty$? **1**

   (b) Let E be a non-measurable set in $\mathbb{R}$. Is $\bigcup\limits_{x\in\mathbb{R}} (E+x)$ Lebesgue measurable? **1**

   (c) Let F be a closed set in $\mathbb{R}$ with $m(F) = 0$. Does it imply that $\text{Int}(F) = \emptyset$? **1**

   (d) Does there exist two non-empty disjoint sets A and B in $\mathbb{R}$ such that $\inf\{|x-y| : x \in A \text{ and } y \in B\} = 0$? **1**

2. Let $f : \mathbb{R} \to [0, \infty]$ be such that $m^*\{x \in \mathbb{R} : f(x) \geq 2^n\} < \frac{1}{2^n}$, whenever $n \in \mathbb{N}$. Show that $\{x \in \mathbb{R} : f(x) = \infty\}$ is Lebesgue measurable. **3**

3. (a) Let D be a dense subset of $\mathbb{R}$. For each $x \in \mathbb{R}$, show that there exists an increasing sequence $x_n \in D$ such that $x_n \to x$. **3**

   (b) Further, deduce that $f : \mathbb{R} \to \bar{\mathbb{R}}$ is a Lebesgue measurable function if and only if $\{x \in \mathbb{R} : f(x) > r\}$ is a Lebesgue measurable set for each $r \in D$. **3**

4. (a) Let F_n be a sequence of closed sets in $\mathbb{R}$ such that $F_n \subset (n, n+1]$ and $F_n \cap F_m = \emptyset$, whenever $m \neq n$. Show that $F = \bigcup\limits_{n=1}^{\infty} F_n$ is a closed set in $\mathbb{R}$. **3**

   (b) Let $E = \bigcup\limits_{n=1}^{\infty} E_n$, where $E_n \in M$ and $E_n \cap E_m = \emptyset$, whenever $m \neq n$. If $m^*(E) < \infty$, then prove that for each $\epsilon > 0$, there exist open set O and closed set F in $\mathbb{R}$ such that $F \subset E \subset O$ and $m(O \setminus F) < \epsilon$. **4**

5. Let $E \subset (0,1)$ be such that $E = \bigcup\limits_{n=2}^{\infty} \left\{ \left(\frac{1}{n-1}, \frac{1}{n}\right) \cap E \right\}$. If E is Lebesgue measurable, then show that $\lim\limits_{n\to\infty} m \left\{ \left(\frac{1}{n}, \frac{1}{n-1}\right) \cap E \right\} = 0$. **3**

6. Let $f : (X, d) \to \mathbb{R}$ be a continuous function. Show that $\{x \in X : f(x) \neq 0\}$ is an open set in the metric space (X, d). **3**

7. Let c_{oo} denote the space of all real sequences having only finitely many non-zero terms. Show that $(c_{oo}, \|\cdot\|_\infty)$ is not an open subset of $(l^1, \|\cdot\|_1)$. **3**

8. For $n \in \mathbb{N}$, define $f_n(t) = te^{-nt^2}$. Show that f_n is a convergent sequence in the space $(C[0,1], \|\cdot\|_\infty)$. **3**

9. For $n \in \mathbb{N}$, write $E = \bigcup_{n=1}^{\infty} \left[n, n + \frac{1}{n^{3/2}}\right]$. Show that $m(E) < \infty$ and $m(E^2) = \infty$, where $E^2 = \{x^2 : x \in E\}$. **3**

10. Let C be the Cantor's ternary set. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in C \setminus \{0\}, \\ x & \text{otherwise.} \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{[0,1]} f dm$. **3**

11. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$
Show that $\int\limits_{[0,1]} f dm < \infty$. **1+3**

12. Let $\varphi : (\mathbb{R}, M, m) \to [0, \infty]$ be a Lebesgue measurable simple function. Define a set function $\nu : M \to [0, \infty]$ by $\nu(E) = \int\limits_E \varphi dm$. Show that $\nu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \nu(E_n)$, whenever E_n is a sequence of pairwise disjoint sets in M. **3**

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