DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA543: Functional Analysis — Quiz III
Instructor: Rajesh Srivastava — November 17, 2020
Time duration: One hour — Maximum Marks: 15

N.B. Answer without proper justification will attract zero mark.

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1. Is it necessary that the graph of a linear functional on an infinite-dimensional Banach space closed? **[1]**

2. Let $(X, \|\cdot\|)$ be Banach space and $B = \{x \in X : \|x\| = 1\}$. If $T \in B(X)$ is non-invertible, does it imply $T(B)$ compact? **[1]**

3. Let X be Banach space and $f_n \in X^*$ be such that $\sum_{n=1}^{\infty} f_n(x)$ is convergent for each $x \in X$. Show that $\left(\frac{\|f_n\|}{n^2}\right) \in \ell^2(\mathbb{N})$. **[3]**

4. Suppose $x \in C[0,1]$. Show that there exists unique $y \in C[0,1]$ such that
$$x(t) = y(t) + \frac{1}{2}\int_0^1 \sin(s+t)x(t)ds.$$
[2]

5. Let X_1 and X_2 be two closed subspaces of a Banach space X such that $X = X_1 \oplus X_2$, where $X_1 \cap X_2 = \{0\}$ and each $x \in X$ has unique representation as $x = x_1 + x_2, ;\ x_i \in X_i, ;\ i = 1, 2$. Show that there exists $k > 0$ such that $\|x_1\| + \|x_2\| \leq k\|x\|$ for all $x \in X$. **[2]**

6. Let X a be Banach space and f belongs to the unit sphere of X^*. For each $x \in X$, prove that $|f(x)| = \inf\{\|x - y\| : y \in \ker f\}$. **[3]**

7. Let $1 \leq p < \infty$, and let $(a_1, a_2, \ldots)$ be sequence of complex numbers such that $\sum_{n=1}^{\infty} a_n x_n$ is convergent for each $(x_1, x_2, \ldots) \in \ell^p(\mathbb{N})$. Show that $(a_1, a_2, \ldots) \in \ell^q(\mathbb{N})$, where $\frac{1}{p} + \frac{1}{q} = 1$. **[3]**

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