DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA543: Functional Analysis	Quiz IV
Instructor: Rajesh Srivastava	November 30, 2020
Time duration: One hour	Maximum Marks: 15

N.B. Answer without proper justification will attract zero mark.

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1. Suppose M be a nowhere dense closed subset of a Hilbert space H. Does it imply $(H \setminus M)^{\perp} = \{0\}$? **[1]**

2. If any pair of points x and y in an inner product space X satisfies $\|x+\alpha y\| = \|x-\alpha y\|$ for each α in the unit circle of $\mathbb{C}$, then x must be orthogonal to y. **[2]**

3. Consider the sequence $f_n \in L^2[0,1]$ defined by

$$f_n(t) = \begin{cases} \sqrt{\frac{n}{1+t}} & \text{if } 0 \leq t < 1/n, \\ 0 & \text{if } 1/n \leq t \leq 1. \end{cases}$$

   Show that $\lim_{n\to\infty} \|f_n\|_2 = 1$ and f_n converges to 0 in the weak topology of $L^2[0,1]$. **[3]**

4. Let $M = \{(x_1, x_2, \ldots) \in \ell^2) : x_1 + 2x_2 = 0\}$. Define a linear functional f on M by $f(x_1, x_2, \ldots) = \frac{3}{2}x_1$. Find a norm preserving extension of f to ℓ^2. **[3]**

5. Let $Y = \text{span}\{(1,0,0),(0,1,0)\},\ y_o = (1,2,1)$ and $Y_1 = \{y + \alpha y_0 : y \in Y, \alpha \in \mathbb{C}\}$. Define a function on Y_1 by $f(y + \alpha y_o) = \alpha$. Show that f is a continuous linear functional on Y_1 and $\|f\| = 1$. **[3]**

6. For $f \in C^1[0,1]$, define a norm on $C^1[0,1]$ by $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Define a linear functional φ on $C^1[0,1]$ by $\varphi(f) = f'(0)$. Show that $\varphi \in (C^1[0,1], \|\cdot\|)^*$ but $\varphi \notin (C^1[0,1], \|\cdot\|_1)^*$ **[3]**

END