**DEPARTMENT OF MATHEMATICS**
**Indian Institute of Technology Guwahati**

MA550: Measure Theory — End Semester Exam
Instructor: Rajesh Srivastava — November 28, 2018
Time duration: Three hours — Maximum Marks: 45

**N.B.** Answer without proper justification will attract zero mark.

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1. (a) Let $(X, S, \mu)$ be a $\sigma$- finite measure space with $\mu(\{x\}) = 0$ for all $x \in X$. Is it possible that $(\mu \times \mu)(\{(x, y) \in X \times X : x = y\}) > 0$? $\boxed{\mathbf{1}}$

   (b) Does there exist a Lebesgue measurable function $f$ on $(\mathbb{R}, M, m)$ such that $\int\limits_E f$ is finite for every $E \in M$ but $f \notin L^1(\mathbb{R}, M, m)$? $\boxed{\mathbf{1}}$

   (c) Does there exist $f \in L^\infty(X, S, \mu)$ such that $\mu(\{x \in X : |f(x)| = \|f\|_\infty\}) = 0$? $\boxed{\mathbf{1}}$

2. Let $\mathcal{A}$ be the monotone class generated by all closed sets in $\mathbb{R}$. If $E$ and $F$ are closed subsets $\mathbb{R}$, then show that $E + F$ belongs to $\mathcal{A}$. $\boxed{\mathbf{3}}$

3. Let $(X, S, \mu)$ be a finite measure space. If for each $n \in \mathbb{N}$, the function $\dfrac{e^n f}{e^n + |f|}$ is $S$-measurable, then show that $\int\limits_X |f| = \lim \int\limits_X \frac{e^n f}{e^n+|f|}$. Does $f$ belong to $L^1(X, S, \mu)$? $\boxed{\mathbf{2+1}}$

4. Let $f : (X, S, \mu) \to (0, \infty)$ be a measurable function. If $\int\limits_E f d\mu = 0$ for some $E \in S$, then prove that $\mu(E) = 0$. $\boxed{\mathbf{3}}$

5. Let $(X, S, \mu)$ be a finite measure space. For a fix $E \in S$, define a sequence $f_n$ of functions on $X$ by $f_n = \begin{cases} \chi_E & \text{if } n \text{ is odd,} \\ 1 - \chi_E & \text{if } n \text{ is even.} \end{cases}$
   If $\mu(E)\mu(E^c) > 0$, then show that $\int\limits_X \underline{\lim} f_n d\mu < \underline{\lim} \int\limits_X f_n d\mu$. $\boxed{\mathbf{3}}$

6. Let $1 \leq p < \infty$ and $f, f_n \in L^p(X, S, \mu)$ be such that $\|f_n - f\|_p \to 0$. For each $\epsilon > 0$, show that $\mu\{x \in X : |f_n(x) - f(x)| > \epsilon\} \to 0$. $\boxed{\mathbf{2}}$

7. Let $1 < p < \infty$ and $f \in L^p(X, S, \mu)$. Show that the series $\sum\limits_{n=1}^{\infty} \mu\{x \in X : |f(x)| \geq n\}$ is convergent. $\boxed{\mathbf{2}}$

8. Let $(X, S, \mu)$ be a $\sigma$- finite measure space. Suppose for each $\epsilon > 0$ there exists some $p > 1$ such that $\|f\|_p < \epsilon$ for every $f \in L^p(X, S, \mu)$. Show that $\mu = 0$. $\boxed{\mathbf{4}}$

9. Let $(X, S, \mu)$ be a finite measure space. Show that $L^\infty(X, S, \mu)$ is a dense subspace of $L^p(X, S, \mu)$ for any $p \geq 1$. $\boxed{\mathbf{3}}$

10. Find the $\lim\limits_{n\to\infty} \int\limits_0^1 \dfrac{nx \sin x}{1 + (nx)^3}\, dx$. $\boxed{\mathbf{3}}$

11. Define a linear functional on $L^1(\mathbb{R}, M, m)$ by $T(f) = \int\limits_{\mathbb{R}} \frac{f(x)}{1+|x|}$. Show that $T$ is bounded and $\|T\| = 1$. $\boxed{\mathbf{1+3}}$

12. For $f \in L^1(\mathbb{R}, M, m,)$ define $F(x) = \int\limits_0^x f(t)dt$. Show that $F \in L^1([0,1], M, m)$ and it satisfies $\|F\|_1 \leq \|f\|_1$. $\boxed{\mathbf{1+3}}$

13. Let $D = \{(x, y) :\in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$ and $f(x, y) = y^2 \sin xy$. Show that $f\chi_D \in L^1(\mathbb{R}^2, M \otimes M, m \times m)$ Further, deduce that $\int\limits_D f\, d(m \times m) = \int\limits_{[0,1]} g dm$, where $g(y) = y(1 - \cos y^2)$. $\boxed{\mathbf{3+2}}$

14. Let $f : (X, S, \mu) \to [0, 1]$ be a measurable function on the finite measure space $(X, S, \mu)$. Show that $A_f = \{(x, y) \in X \times [0, 1] : f(x) \leq y\}$ is $S \otimes \mathcal{B}(\mathbb{R})$- measurable. Further derive that $(\mu \times m)(A_f) = \mu(X) - \int\limits_X f(x)d\mu(x)$. $\boxed{\mathbf{3+2}}$

**END**