DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA550: Measure Theory	Mid Semester Exam
Instructor: Rajesh Srivastava	September 20, 2018
Time duration: Two hours	Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) Does there exist a Lebesgue measurable set $A \subset \mathbb{R}$ such that $m(A) = 0$ and $m(\text{boundary}(A)) = \infty$? **[1]**

   (b) Let C be the Cantor set in $[0,1]$ and $a, b \in \mathbb{R}$ with $a < b$. Whether the set $C + (a, b)$ is Borel measurable? **[1]**

   (c) Let $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 1$. Does it imply that the function f is Borel measurable on $\mathbb{R}$? **[1]**

   (d) Let A be subset of $\mathbb{R}$ with $m^*(A) < \infty$. Does it imply that $m^*(A^2) < \infty$? **[1]**

2. $\tilde{M}$ be the class of all Lebesgue measurable subset of $[0,1]$. If $N \notin \tilde{M}$. Prove/disprove $N \cap (\mathbb{R} \setminus \mathbb{Q}) \in \tilde{M}$. **[2]**

3. Let $E \subset [0,1]$ be Lebesgue measurable and $m(E) > 0$. For $r_n \in [-1,1] \cap \mathbb{Q}$, let $E_n = E + r_n$. Show that all of E_n's cannot be pairwise disjoint. Further, deduce that there exist $x, y \in E$ such that $x - y \in \mathbb{Q}$. **[3]**

4. Let A and B be subsets of $[0,1]$ which satisfy $m^*(A \cup B) = m^*(A) + m^*(B)$. If $A \triangle B$ is Lebesgure measurable then prove that A and B are Lebesgure measurable. **[4]**

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f = 0$ except on set of Lebesgue measure zero. Show that f is identically zero on $\mathbb{R}$. **[3]**

6. Let (X, S, μ) be a finite measure space and $f : X \to \bar{\mathbb{R}}$ be an almost finite S-measurable function. Prove that for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu\{x \in X : |f(x)| > n_0\} < \epsilon$. **[3]**

7. Let $f : (\mathbb{R}, M, m) \to [0, \infty]$ be such that for each $\epsilon > 0$ there exists a Lebesgue measurable set $E \subset \mathbb{R}$ with $m(E) < \epsilon$ and f is continuous on $\mathbb{R} \setminus E$. Show that f is a Lebesgue mesurable function. **[4]**

8. Let $E \subset \mathbb{R}$ be Lebesgue measurable and $m(E) = \infty$. Define a function $f : \mathbb{R} \to \bar{\mathbb{R}}$ by $f(x) = m(E \cap (-\infty, x))$. Show that f is a Borel measurable function. **[4]**

9. Let $g : [0,1] \to [0,2]$ be a bijection with $m(g(C)) = 1$, where C is the Cantor set. Construct a Lebesgue measurable function f on $[0,1]$ such that $f \circ g^{-1}$ is not Lebesgue measurable. **[3]**

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