# Assignment 1

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If $(X, \|\cdot\|)$ is a normed linear space such that $\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in X$, then there must exist an inner product on $X$ which induces the norm $\|\cdot\|$ on $X$.
   (b) Every orthonormal set in a Hilbert space $H$ must be closed in $H$.
   (c) If a proper subspace $M$ of a Hilbert space $H$ contains an orthonormal basis of $H$, then $M$ cannot be a Banach space in the induced norm.
   (d) In any infinite dimensional Hilbert space, there exists a convergent series which is not absolutely convergent.
   (e) If $(x_n)$ is a sequence in a Hilbert space $H$ such that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$, then the series $\sum_{n=1}^{\infty} x_n$ must converge in $H$.
   (f) If $(u_n)$ is an orthonormal sequence in a Hilbert space $H$ and if $x \in H$, then the series $\sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$ must converge in $H$ but not necessarily to $x$.
   (g) If $(x_n)$ is an unbounded sequence in a Hilbert space $H$, then there must exist $x \in H$ such that the sequence $(\langle x, x_n \rangle)$ is unbounded.
   (h) If in a Hilbert space $H$, every weakly convergent sequence is norm convergent, then $H$ must be separable.
   (i) If $(x_n)$ is a sequence in a Hilbert space $H$ such that $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{w} x \in H$, then it is necessary that $\|x\| \leq 1$.
   (j) Suppose $X$ is a Banach space and $f_n \in X^*$ and $f_n \xrightarrow{w^*} 0$. Then $f_n$ is necessarily a bounded sequence in $X^*$.

2. Let $X$ be an inner product space and let $x$, $y \in X$. Prove that
   (a) $\|x+y\|\|x-y\| \leq \|x\|^2 + \|y\|^2$.
   (b) if $\lambda > 0$, then $|\langle x, y \rangle| \leq \lambda \|x\|^2 + \frac{1}{4\lambda}\|y\|^2$.
   (c) if $\delta = \inf\{\|\alpha x + y\| : \alpha \in \mathbb{K}\}$, then $|\langle x, y \rangle|^2 \leq (\|y\|^2 - \delta^2)\|x\|^2$.
   (d) $\Big| \|x\| - \|y\| \Big| = \|x - y\|$ iff $tx = sy$ for some $t, s \geq 0$ with $(t, s) \neq (0, 0)$.

3. Let $X$ be an inner product space and let $v, w \in X$ such that $\|v\|\|w\| < 1$. Show that for each $y \in X$, there exists a unique $x \in X$ such that $y = x + \langle x, v \rangle w$.

4. Show that it is impossible to define an inner product on $X$ which induces the norm $\|\cdot\|$ on $X$, where $(X, \|\cdot\|)$ is
   (a) $(c_{00}, \|\cdot\|_\infty)$ (b) $(C[a, b], \|\cdot\|_\infty)$ (c) $\mathcal{B}((\ell^2, \|\cdot\|_2))$ with the usual norm

5. Let $X$ be a normed linear space such that every two dimensional subspace of $X$ is an inner product space. Show that $X$ is an inner product space.

6. Consider the following three conditions regarding the points $x$, $y$ in an inner product space.
   (a) $\|x + \alpha y\| = \|x - \alpha y\|$ for all $\alpha \in \mathbb{K}$
   (b) $\|x + \alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{K}$
   (c) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

Prove that each of (a) and (b) is a necessary and sufficient condition for $x \perp y$. Do you have a similar statement for (c)?

7. Let $Y$ be a closed subspace of an inner product space $X$. Show that there exists an inner product on the quotient space $X/Y$ which induces the quotient norm on $X/Y$.

8. Let $M$ be a closed subspace of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$.
If $\langle x_1 + M, x_2 + M \rangle_0 = \langle x_1 - P_M x_1, x_2 - P_M x_2 \rangle$ for all $x_1, x_2 \in H$, then show that $\langle \cdot, \cdot \rangle_0$ is an inner product on the quotient space $H/M$ which induces the quotient norm on $H/M$.

9. If $M$ and $N$ are closed subspaces of a Hilbert space, then show that $(M \cap N)^\perp = \overline{M^\perp + N^\perp}$.

10. Let $X$ be an inner product space and let $x \in X$. If $M = \{z \in X : \langle x, z \rangle = 0\}$, then determine $M^\perp$ and $M^{\perp\perp}$.

11. Let $S$ be a nonempty subset of a Hilbert space $H$. Show that $S^{\perp\perp} = \overline{\text{span}(S)}$.
Hence deduce that $\text{span}(S)$ is dense in $H$ iff $S^\perp = \{0\}$.

12. Let $M$ be a subspace of an inner product space $X$ and let $x \in X$. Prove that $x \perp M$ iff $\|x\| \leq \|x + y\|$ for all $y \in M$.

13. Let $M$ be a closed subspace of a Hilbert space $H$ and let $x \in H \setminus M$. Prove that $d(x, M) = \sup\{|\langle x, y \rangle| : y \in M^\perp, \ \|y\| \leq 1\}$.

14. Let $H$ be a Hilbert space. Let $M \subset H$ and let $T : H \to H$ be linear such that $Tx \in M$ and $x - Tx \in M^\perp$ for each $x \in H$. Show that $M$ is a closed subspace of $H$.

15. Let $M$ be a nonempty subset of a Hilbert space $H$ and let $z \in H$. Show that there exists $u \in M^\perp$ such that $\langle x, z \rangle = \langle x, u \rangle$ for all $x \in M^\perp$.

16. If $\omega = e^{2\pi i/3}$, then which point in the subspace span $\{(1, \omega, \omega^2), (1, \omega^2, \omega)\}$ of the Hilbert space $\mathbb{C}^3$ is nearest to the point $(1, -1, 1)$?

17. Let $C$ be a nonempty convex subset of an inner product space $X$ and let $x \in X$. Prove that for $y \in C$, $d(x, C) = \|x - y\|$ iff $\text{Re}\langle x - y, z - y \rangle \leq 0$ for all $z \in C$.

18. Let $C$ be a nonempty convex set in an inner product space $X$ and let $(x_n)$ be a sequence in $C$ such that $\lim_{n \to \infty} \|x_n\| = \inf_{x \in C} \|x\|$. Show that $(x_n)$ is a Cauchy sequence in $X$.

19. Let $M$ be a closed subspace of a Hilbert space $H$. If $x \in M$ and if $(x_n)$ is a sequence in $M$, then show that $x_n \xrightarrow{w} x$ in $H$ iff $x_n \xrightarrow{w} x$ in $M$.

20. Let $(x_n)$ be a sequence in a Hilbert space $H$ such that for each $x \in H$, the sequence $(\langle x_n, x \rangle)$ converges in $\mathbb{K}$. Show that there exists $y \in H$ such that $x_n \xrightarrow{w} y$ in $H$.

21. Using Riesz representation theorem, show that $\left\{(x_n) \in \ell^2 : \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}} x_n = 0\right\}$ is not a closed subset of the Hilbert space $\ell^2$.

22. Let $\{u_1, ..., u_n\}$ be an orthonormal set in a Hilbert space $H$. Prove that $\|x - \sum\limits_{i=1}^{n} \alpha_i u_i\| \geq \|x - \sum\limits_{i=1}^{n} \langle x, u_i \rangle u_i\|$ for all $x \in H$ and for all $(\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$.

23. Let $(T_n)$ be a Cauchy sequence in $\mathcal{B}(X)$, where $X$ is an inner product space. Let $y \in X$ and let $f_n(x) = \langle T_n x, y \rangle$ for all $x \in X$. Show that $(f_n)$ is a convergent sequence in $X^*$.

24. Let $(u_n)$ be an orthonormal sequence in an inner product space $X$ and let $(\alpha_n)$ be a sequence in $\mathbb{K}$. If $s_n = \sum\limits_{i=1}^{n} \alpha_i u_i$ for all $n \in \mathbb{N}$, then show that $(s_n)$ is a Cauchy sequence in $X$ iff $(\alpha_n) \in \ell^2$.

25. Let $H$ be an infinite dimensional Hilbert space. Show that no orthonormal basis of $H$ can be a Hamel basis of $H$.

26. Let $f(x) = \int\limits_0^1 t x(t)\, dt$ for all $x \in C[0,1]$. Show that $f \in (C[0,1], \| \cdot \|_2)^*$ and find $\|f\|$.

27. Let $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space $H$ and let $f \in H^*$. Prove that $y = \sum\limits_{n=1}^{\infty} \overline{f(u_n)} u_n$ is the unique element in $H$ such that $f(x) = \langle x, y \rangle$ for all $x \in H$ and that $\|f\|^2 = \sum\limits_{n=1}^{\infty} |f(u_n)|^2$.

28. Let $(H, \| \cdot \|)$ be a separable Hilbert space with an orthonormal basis $\{u_n : n \in \mathbb{N}\}$. If $\|x\|_0 = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} |\langle x, u_n \rangle|$ for all $x \in H$, then show that $\| \cdot \|_0$ is a norm on $H$ which is not equivalent to $\| \cdot \|$.

29. Let $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space $H$ and let $\{v_n : n \in \mathbb{N}\}$ be an orthonormal set in $H$ such that $\sum\limits_{n=1}^{\infty} \|u_n - v_n\|^2 < 1$. Show that $\{v_n : n \in \mathbb{N}\}$ is an orthonormal basis of $H$.

30. Find $\min\limits_{a,b,c \in \mathbb{R}} \int\limits_{-1}^{1} |x^3 - a - bx - cx^2|^2\, dx$.

31. Give an example to show that the range of an one-one continuous linear map from a Hilbert space $H$ to itself need not be closed in $H$.

32. If $\{u_n : n \in \mathbb{N}\}$ is an (countably infinite) orthonormal basis of a Hilbert space $H$, then show that there exists a discontinuous linear map $T : H \to H$ such that $T u_n = 0$ for all $n \in \mathbb{N}$.

33. Let $H$ be a Hilbert space and let $(T_n)$ be a sequence in $\mathcal{B}(H)$ such that for each $x, y \in H$, $\lim\limits_{n \to \infty} \langle T_n x, y \rangle$ exists in $\mathbb{K}$. Show that $\sup\{\|T_n\| : n \in \mathbb{N}\} < \infty$.

34. If $H$ is a non-zero Hilbert space, then show that there cannot exist $T, S \in \mathcal{B}(H)$ such that $TS - ST = I$, where $I$ denotes the identity operator on $H$.

35. Let $X$ and $Y$ be two normed linear spaces. Suppose $T : X \to Y$ is a linear map that sends every weakly convergence sequence a weakly convenance sequence. Show that $T$ is bounded.

36. Suppose $X$ is a Banach space and $x, x_n \in X$. Prove that $x_n \xrightarrow{w} x$ if and only if $\{\|x_n\|\}$ is bounded and $f(x_n) \to f(x)$ for each $f \in S$, where $\overline{\text{Span } S} = X^*$.

37. A sequence $\{f_n\}$ in $X^*$ is weak$^*$ convergent if and only if $\{\|f_n\|\}$ is bounded and $\{f_n(x)\}$ is a Cauchy sequence for each $x \in S$, where $\overline{\text{Span } S} = X$.

38. Suppose $X$ is a Banach space and $x, x_n \in X$. Prove that $x_n \xrightarrow{w} x$ if and only if $\{\|x_n\|\}$ is bounded and $f(x_n) \to f(x)$ for each $f \in S$, where $\overline{\text{Span } S} = X^*$.

39. A sequence $\{f_n\}$ in $X^*$ is weak$^*$ convergent if and only if $\{\|f_n\|\}$ is bounded and $\{f_n(x)\}$ is a Cauchy sequence for each $x \in S$, where $\overline{\text{Span } S} = X$.