DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA641: Operator Theory in Hilbert Spaces — EndSem
Instructor: Rajesh Srivastava — June 8, 2020
Time duration: Four hours — Maximum Marks: 40

N.B. Answer without proper justification will attract zero mark.

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1. Let $T : l^\infty \to l^\infty$ be define by $T(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \ldots)$. Find a non-zero proper separable invartiant subspace of T. **[2]**
2. Let $g \in L^\infty(\mathbb{R})$ and T on $L^2(\mathbb{R})$ be defined by $Tf = gf$. Show that $\|T\| = \|g\|_\infty$. Further, derive that $\sigma_{com}(T) = \emptyset$. **[5]**
3. Let $T : L^2[0,1] \to L^2[0,1]$ be defined by $Tf(t) = f(\frac{t}{2})$. Find the adjoint T^* of T. Show that $0 \notin \sigma_c(T) \cup \sigma_p(T^*)$. **[4]**
4. Let T be a positive compact operator on a complex Hilbert space H. Show that there exists a positive compact operator S on H such that $S^2 = T$. **[5]**
5. Let $\{e_n\}$ be an orthonormal basis for a complex Hilbert space H. If $\lambda_n \in \mathbb{R}$ be such that $\lambda_n \to 0$. Then show that there exists a unique self-adjoint compact operator T such that $Te_n = \lambda_n e_n$. **[5]**
6. If $T : l^2 \to l^2$ is define by $T(x_1, x_2, x_3, x_4, \ldots) = (x_1 + x_2, x_2, x_3 + x_4, x_4, \ldots)$. Then find $\rho(T), \sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$. **[5]**
7. Let H be separable Hilbert space. Show that for every closed set F in $\mathbb{C}$, there exists a sequence T_n of compact operators on H such that $F = \bigcup_{n=1}^{\infty} \sigma(T_n)$. **[5]**
8. Let T be a nonzero bounded operator on a complex Hilbert space H. Does it imply $\sigma_{com}(T) \subset \{\lambda \in \mathbb{C} : |\lambda| < \|T\|\}$? **[3]**
9. Suppose $g \in L^\infty(\mathbb{R})$. Define an operator T on $L^1(\mathbb{R})$ by $Tf = gf$. Find a non-zero proper invariant subspace of T. **[3]**
10. Let $T \in \mathcal{B}(l^2)$ be a normal operator. Show that $\sigma_p(T)$ of T is countable. **[3]**

END