DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA641: Operator Theory in Hilbert Spaces	MidSem
Instructor: Rajesh Srivastava	March 1, 2020
Time duration: Two hours	Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) Whether weak topology on a separable Hilbert space is metrizable? **[1]**

   (b) Let H be a Hilbert space. Let T_n be sequence of compact operators on H and $T \in \mathcal{B}(H)$. Suppose $\|T_n(x) - T(x)\| \to 0$ for all $x \in H$. Is it necessary T a compact operator? **[1]**

   (c) Suppose $T_n : l_2 \to l_2$ be a sequence of completely continuous linear transformations. If $T_n \to T$ in $\mathcal{B}(l_2)$, does it imply that T is compact? **[1]**

   (d) Let P_1 and P_2 be orthogonal projections on a Hilbert space H satisfying $P_1P_2 = 0$. Does it imply that $P_2P_1 = 0$? **[1]**

2. Let H_1 and H_2 be Hilbert space and let $T \in \mathcal{B}(H_1, H_2)$. Show that T^* (the adjoint operator of T) is one-one if and only if $\overline{T(H_1)} = H_2$. **[3]**

3. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space H. If $T : H \to H$ is linear map with $(Te_n) \in l^2$. Show that T is bounded. Whether T is a compact operator? **[5]**

4. Find $\min\limits_{a,b\in\mathbb{R}} \int_{-1}^{1} |x^2 - a - bx|^2 \, dx$. **[2]**

5. Let H_1 and H_2 be Hilbert spaces and let $T \in \mathcal{B}(H_1, H_2)$ be such that $\dim(T(H_1) = 1$. Show that there exist $y \in H_1$ and $z \in H_2$ such that $T(x) = \langle x, y\rangle z$ for all $x \in H_1$. Whether y and z are unique? **[3]**

6. Let P_1 and P_2 be orthogonal projections on a Hilbert space H. Show that $\|P_1 + P_2\| = \|P_1\| + \|P_2\|$ if and only if $P_1P_2 = 0$. **[2]**

7. Suppose $T : l^2 \to l^2$ be such that $T^2 = 0$. Show by an example that T need not be a compact operator. **[2]**

8. Let X be a Banach space. Suppose $S, T \in \mathcal{B}(X)$ be such that $ST - TS = I$, where I is the identity operator in $\mathcal{B}(X)$. Show that there exists some $n \in \mathbb{N}$ such that $T^n = 0$. Conclude that if P and Q are linear transformation on X, then $PQ - QP = I$ will holds if one of the P and Q is not bounded. **[4]**

9. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space H. If $T : H \to H$ is linear transfrmation that defined by $T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n$ for some sequence $(\lambda_n) \subset \mathbb{C}$. Show that T is continuous if and only if (λ_n) is bounded. Further, if $\lambda_n \to 0$, does it imply that T is a compact operator? **[5]**

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