MA 101 (Mathematics I)

Practice Problem Set - 1

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If both (x_n) and (y_n) are unbounded sequences in $\mathbb{R}$, then the sequence (x_ny_n) cannot be convergent.
   (b) If both (x_n) and (y_n) are increasing sequences in $\mathbb{R}$, then the sequence (x_ny_n) must be increasing.
   (c) If (x_n), (y_n) are sequences in $\mathbb{R}$ such that (x_n) is convergent and (y_n) is not convergent, then the sequence $(x_n + y_n)$ cannot be convergent.
   (d) A monotonic sequence (x_n) in $\mathbb{R}$ is convergent iff the sequence (x_n^2) is convergent.
   (e) If (x_n) is an unbounded sequence of nonzero real numbers, then the sequence $(\frac{1}{x_n})$ must converge to 0.
   (f) If $x_n = (1 - \frac{1}{n}) \sin \frac{n\pi}{2}$ for all $n \in \mathbb{N}$, then the sequence (x_n) is not convergent although it has a convergent subsequence.
   (g) If both the series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ of real numbers are convergent, then the series $\sum\limits_{n=1}^{\infty} x_ny_n$ must be convergent.
   (h) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in \mathbb{Q}$, then it is necessary that $f(x) > 0$ for all $x \in \mathbb{R}$.
   (i) There exists a continuous function from $(0, 1)$ onto $(0, \infty)$.
   (j) There exists a continuous function from $[0, 1]$ onto $(0, 1)$.
   (k) There exists a continuous function from $(0, 1)$ onto $[0, 1]$.
   (l) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, then there must exist $c \in \mathbb{R}$ such that $f(c) = c$.
   (m) If both $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous at 0, then the composite function $g \circ f : \mathbb{R} \to \mathbb{R}$ must be continuous at 0.
   (n) If $f : \mathbb{R} \to \mathbb{R}$ is not differentiable at $x_0 \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is not differentiable at $f(x_0)$, then $g \circ f : \mathbb{R} \to \mathbb{R}$ cannot be differentiable at x_0.
   (o) If $f : \mathbb{R} \to \mathbb{R}$ is such that $\lim\limits_{h \to 0} \frac{f(x+h)-f(x-h)}{h}$ exists (in $\mathbb{R}$) for every $x \in \mathbb{R}$, then f must be differentiable on $\mathbb{R}$.

2. Using the definition of convergence of sequence, examine whether the following sequences are convergent.
   (a) $(n + \frac{3}{2})$
   (b) $((-1)^n \frac{3}{n+2})$
   (c) $((-1)^n (1 - \frac{1}{n}))$
   (d) $(\frac{3n^2+\sin n-4}{2n^2+3})$
   (e) $(\frac{2\sqrt{n}+3n}{2n+3})$

3. Examine whether the sequences (x_n) defined as below are convergent. Also, find their limits if they are convergent.
   (a) $x_n = (a^n + b^n + c^n)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, where a, b, c are distinct positive real numbers.
   (b) $x_n = \frac{1-n+(-1)^n}{2n+1}$ for all $n \in \mathbb{N}$.
   (c) $x_n = \frac{n^k}{\alpha^n}$, where $|\alpha| > 1$ and $k > 0$.
   (d) $x_n = \frac{p(n)}{2^n}$ for all $n \in \mathbb{N}$, where $p(x)$ is a polynomial in the real variable x of degree 5.
   (e) $x_n = \frac{3.5.7.\cdots.(2n+1)}{2.5.8.\cdots.(3n-1)}$ for all $n \in \mathbb{N}$.
   (f) $x_n = \frac{1}{n} \sin^2 n$ for all $n \in \mathbb{N}$.
   (g) $x_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$ for all $n \in \mathbb{N}$.
   (h) $x_n = \frac{n}{n^3+1} + \frac{2n}{n^3+2} + \cdots + \frac{n^2}{n^3+n}$ for all $n \in \mathbb{N}$.

(i) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n+1}}$ for all $n \in \mathbb{N}$.
(j) $x_n = \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}\right)$ for all $n \in \mathbb{N}$.
(k) $x_n = \left(\frac{\sin n + \cos n}{3}\right)^n$ for all $n \in \mathbb{N}$.
(l) $x_n = \sqrt{4n^2+n} - 2n$ for all $n \in \mathbb{N}$.
(m) $x_n = \sqrt{n^2+n} - \sqrt{n^2+1}$ for all $n \in \mathbb{N}$.
(n) $x_1 = 1$ and $x_{n+1} = 1 + \sqrt{x_n}$ for all $n \in \mathbb{N}$.
(o) $x_1 = 4$ and $x_{n+1} = 3 - \frac{2}{x_n}$ for all $n \in \mathbb{N}$.
(p) $x_1 = 0$ and $x_{n+1} = \sqrt{6 + x_n}$ for all $n \in \mathbb{N}$.
(q) $x_1 > 1$ and $x_{n+1} = \sqrt{x_n}$ for all $n \in \mathbb{N}$.

4. Let (x_n), (y_n) be sequences in $\mathbb{R}$ such that $x_n \to x \in \mathbb{R}$ and $y_n \to y \in \mathbb{R}$. Show that $\lim_{n\to\infty} \max\{x_n, y_n\} = \max\{x, y\}$.

5. If a sequence (x_n) of positive real numbers converges to $\ell \in \mathbb{R}$, then show that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\ell}$.

6. Let (x_n) be a convergent sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = \ell \neq 0$. Show that there exists $n_0 \in \mathbb{N}$ such that $x_n \neq 0$ for all $n \geq n_0$.

7. If $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ for all $n \in \mathbb{N}$, then show that the sequence (x_n) is convergent.

8. Show that the sequences (x_n) in $\mathbb{R}$ defined as below are Cauchy (and hence convergent). Also, find their limits.
(a) $x_1 = 1$ and $x_{n+1} = \frac{2+x_n}{1+x_n}$ for all $n \in \mathbb{N}$.
(b) $x_1 > 0$ and $x_{n+1} = 2 + \frac{1}{x_n}$ for all $n \in \mathbb{N}$.

9. Examine whether the sequence (x_n) has a convergent subsequence, where for each $n \in \mathbb{N}$,
(a) $x_n = (-1)^n n^2$ (b) $x_n = (-1)^n \frac{5n \sin^3 n}{3n-2}$.

10. If $a, b \in \mathbb{R}$, then show that the series $a + (a+b) + (a+2b) + \cdots$ is not convergent unless $a = b = 0$.

11. Examine whether the following series are convergent.
(a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^n}$
(c) $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$
(d) $\sum_{n=1}^{\infty} \sqrt{\frac{2n^2+3}{5n^3+1}}$
(e) $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$
(f) $\sum_{n=1}^{\infty} ((n^3+1)^{\frac{1}{3}} - n)$
(g) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$
(h) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$
(i) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+1}}{n+1}$

12. Find all $x \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n!}$ is convergent.

13. Find all $x \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} \frac{(x+2)^n}{3^n\sqrt{2n+1}}$ is convergent.

14. Show that the series $\sum\limits_{n=1}^{\infty} \frac{a^n}{a^n+n}$ is convergent if $0 < a < 1$ and is not convergent if $a > 1$.

15. If $0 < x_n < \frac{1}{2}$ for all $n \in \mathbb{N}$ and if the series $\sum\limits_{n=1}^{\infty} x_n$ converges, then show that the series $\sum\limits_{n=1}^{\infty} \frac{x_n}{1-x_n}$ converges.

16. Let (x_n), (y_n) be sequences in $\mathbb{R}$ such that $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$. Find out (with justification) the true statement(s) from the following.
    (a) If the series $\sum\limits_{n=1}^{\infty} y_n$ converges, then the series $\sum\limits_{n=1}^{\infty} x_n$ must converge.
    (b) If the series $\sum\limits_{n=1}^{\infty} x_n$ converges, then the series $\sum\limits_{n=1}^{\infty} y_n$ must converge.
    (c) If the series $\sum\limits_{n=1}^{\infty} y_n$ converges absolutely, then the series $\sum\limits_{n=1}^{\infty} x_n$ must converge absolutely.
    (d) If the series $\sum\limits_{n=1}^{\infty} x_n$ converges absolutely, then the series $\sum\limits_{n=1}^{\infty} y_n$ must converge absolutely.

17. If a series $\sum\limits_{n=1}^{\infty} x_n$ is convergent but the series $\sum\limits_{n=1}^{\infty} x_n^2$ is not convergent, then show that the series $\sum\limits_{n=1}^{\infty} x_n$ is conditionally convergent.

18. Examine whether the following series are conditionally convergent.
    (a) $\sum\limits_{n=1}^{\infty} (-1)^n(\sqrt{n^2+1} - n)$
    (b) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n^2+(-1)^n}$
    (c) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{a^2+n}{n^2}$, where $a \in \mathbb{R}$

19. Find all $x \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} \frac{\log(n+1)}{\sqrt{n+1}}(x-5)^n$ is convergent.

20. Find all $x \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n(x+3)^n}{n5^n}$ is conditionally convergent.

21. Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $|f(x)| \leq |g(x)|$ for all $x \in \mathbb{R}$. If g is continuous at 0 and $g(0) = 0$, then show that f is continuous at 0.

22. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{1}{x}\sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
    Examine whether f is continuous at 0.

23. Give an example (with justification) of a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every point of $\mathbb{R}$ but $|f| : \mathbb{R} \to \mathbb{R}$ is continuous.

24. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x) = x^2 + 5$ for all $x \in \mathbb{Q}$. Find $f(\sqrt{2})$.

25. Evaluate $\lim\limits_{n\to\infty} \sin((2n\pi + \frac{1}{2n\pi})\sin(2n\pi + \frac{1}{2n\pi}))$.

26. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(0) > f(1) < f(2)$. Show that f is not one-one.

27. Let $f : [0,1] \to [0,1]$ be continuous. Show that there exists $c \in [0,1]$ such that $f(c) + 2c^5 = 3c^7$.

28. Show that there exists $c \in \mathbb{R}$ such that $c^{179} + \frac{163}{1+c^2+\sin^2 c} = 119$.

29. Let $f, g : [-1,1] \to \mathbb{R}$ be continuous such that $|f(x)| \le 1$ for all $x \in [-1,1]$ and $g(-1) = -1$, $g(1) = 1$. Show that there exists $c \in [-1,1]$ such that $f(c) = g(c)$.

30. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that
(a) if n is odd, then there exists unique $y \in \mathbb{R}$ such that $y^n = x$.
(b) if n is even and $x > 0$, then there exists unique $y > 0$ such that $y^n = x$.

31. If $f : [0,1] \to \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in [0,1]$, then show that there exists $\alpha > 0$ such that $f(x) > \alpha$ for all $x \in [0,1]$.

32. Give an example of each of the following.
(a) A function $f : [0,1] \to \mathbb{R}$ which is not bounded.
(b) A continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ which does not attain $\sup\{f(x) : x \in \mathbb{R}\}$ as well as $\inf\{f(x) : x \in \mathbb{R}\}$.
(c) A continuous and bounded function $f : (0,1) \to \mathbb{R}$ which attains both $\sup\{f(x) : x \in (0,1)\}$ and $\inf\{f(x) : x \in (0,1)\}$.

33. If $f(x) = x \sin x$ for all $x \in \mathbb{R}$, then show that $f : \mathbb{R} \to \mathbb{R}$ is neither bounded above nor bounded below.

34. Let p be an nth degree polynomial with real coefficients in one real variable such that $n (\neq 0)$ is even and $p(0) \cdot p^{(n)}(0) < 0$. Show that p has at least two real zeroes.

35. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at 0 and let $g(x) = xf(x)$ for all $x \in \mathbb{R}$. Show that $g : \mathbb{R} \to \mathbb{R}$ is differentiable at 0.

36. Let $\alpha > 1$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x)| \le |x|^\alpha$ for all $x \in \mathbb{R}$. Show that f is differentiable at 0.

37. Let $f(x) = x^2|x|$ for all $x \in \mathbb{R}$. Examine the existence of $f'(x)$, $f''(x)$ and $f'''(x)$, where $x \in \mathbb{R}$.

38. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 |\cos \frac{\pi}{x}| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
Examine whether f is differentiable (i) at 0 (ii) on $(0,1)$.

39. Examine whether $f : \mathbb{R} \to \mathbb{R}$, defined as below, is differentiable at 0.
(a) $f(x) = \begin{cases} \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$
(b) $f(x) = \begin{cases} \frac{1}{4^n} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

40. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at 0 and $f(0) = f'(0) = 0$. Show that $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = \begin{cases} f(x) \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$ is differentiable at 0.

41. Let $f(x) = x^3 + x$ and $g(x) = x^3 - x$ for all $x \in \mathbb{R}$. If f^{-1} denotes the inverse function of f and if $(g \circ f^{-1})(x) = g(f^{-1}(x))$ for all $x \in \mathbb{R}$, then find $(g \circ f^{-1})'(2)$.

42. If $a, b, c \in \mathbb{R}$, then show that the equation $4ax^3 + 3bx^2 + 2cx = a + b + c$ has at least one root in $(0, 1)$.

43. If $a_0, a_1, ..., a_n \in \mathbb{R}$ satisfy $\frac{a_0}{1.2} + \frac{a_1}{2.3} + \cdots + \frac{a_n}{(n+1)(n+2)} = 0$, then show that the equation $a_0 + a_1 x + \cdots + a_n x^n = 0$ has at least one root in $[0, 1]$.

44. Show that the equation $|x^{10} - 60x^9 - 290| = e^x$ has at least one real root.

45. Find the number of (distinct) real roots of the following equations.
(a) $x^2 = \cos x$
(b) $e^{2x} + \cos x + x = 0$

46. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable such that $f(0) = 0$, $f'(0) > 0$ and $f''(x) > 0$ for all $x \in \mathbb{R}$. Show that the equation $f(x) = 0$ has no positive real root.

47. Show that between any two (distinct) real roots of the equation $e^x \sin x = 1$, there exists at least one real root of the equation $e^x \cos x + 1 = 0$.

48. Let $f(x) = 3x^5 - 2x^3 + 12x - 8$ for all $x \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}$ is one-one and onto.

49. Show that
(a) $\frac{x-1}{x} < \log x < x - 1$ for all $x(\neq 1) > 0$.
(b) $1 + x < e^x < 1 + xe^x$ for all $x(\neq 0) \in \mathbb{R}$.
(c) $2 \sin x + \tan x > 3x$ for all $x \in (0, \frac{\pi}{2})$.
(d) $(1 + x)^\alpha \geq 1 + \alpha x$ for all $x \geq -1$ and for all $\alpha > 1$.

50. Determine all the differentiable functions $f : [0, 1] \to \mathbb{R}$ satisfying the conditions
(a) $f(0) = 0$, $f(1) = 1$ and $|f'(x)| \leq \frac{1}{2}$ for all $x \in [0, 1]$.
(b) $f(0) = 0$, $f(1) = 1$ and $|f'(x)| \leq 1$ for all $x \in [0, 1]$.

51. Let $f : [0, 2] \to \mathbb{R}$ be differentiable and $f(0) = f(1) = 0$, $f(2) = 3$. Show that there exist $a, b, c \in (0, 2)$ such that $f'(a) = 0$, $f'(b) = 3$ and $f'(c) = 1$.

52. Evaluate the following limits.
(a) $\lim_{x \to 0} (\frac{1}{\sin x} - \frac{1}{x})$
(b) $\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x}$
(c) $\lim_{x \to \infty} x(\log(1 + \frac{x}{2}) - \log \frac{x}{2})$
(d) $\lim_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$
(e) $\lim_{x \to \infty} \frac{2x + \sin 2x + 1}{(2x + \sin 2x)(\sin x + 3)^2}$

53. If $f : (0, \infty) \to (0, \infty)$ is differentiable at $a \in (0, \infty)$, then evaluate $\lim_{x \to a} \left(\frac{f(x)}{f(a)} \right)^{\frac{1}{\log x - \log a}}$.

54. Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x(\neq 0) \in \mathbb{R}, \\ 1 & \text{if } x = 0. \end{cases}$
Examine whether $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

55. Using Taylor's theorem, show that
(a) $|\sqrt{1+x} - (1 + \frac{x}{2} - \frac{x^2}{8})| \leq \frac{1}{2}|x|^3$ for all $x \in (-\frac{1}{2}, \frac{1}{2})$.
(b) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} > \cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ for all $x \in (0, \pi)$.
(c) $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for all $x \in (0, \pi)$.

56. Find the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} n! x^n$.

57. Find the interval of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$.

58. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If there is a partition P of $[a, b]$ such that $L(f, P) = U(f, P)$, then show that f is a constant function.

59. Evaluate the following limits.
(a) $\lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=1}^{n} \sqrt{n^2 - k^2}$
(b) $\lim\limits_{n\to\infty} \frac{1}{n}[(n+1)(n+2)\cdots(n+n)]^{\frac{1}{n}}$
(c) $\lim\limits_{x\to 0} \frac{x}{1-e^{x^2}} \int\limits_0^x e^{t^2}\, dt$
(d) $\lim\limits_{n\to\infty} \left(\frac{1^8 + 3^8 + \cdots + (2n-1)^8}{n^9} \right)$

60. If $f : [-1, 1] \to \mathbb{R}$ is continuously differentiable, then evaluate $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f'(\frac{k}{3n})$.

61. Show that
(a) $\frac{\pi^2}{9} \leq \int\limits_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x}\, dx \leq \frac{2\pi^2}{9}$.
(b) $\frac{\sqrt{3}}{8} \leq \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{x}\, dx \leq \frac{\sqrt{2}}{6}$.

62. If $f : [a, b] \to \mathbb{R}$ is continuous, then show that there exists $c \in [a, b]$ such that $\int\limits_a^b f(x)\, dx = (b - a)f(c)$.
(This result is called the mean value theorem of Riemann integrals.)

63. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous and let $g(x) \geq 0$ for all $x \in [a, b]$. Show that there exists $c \in [a, b]$ such that $\int\limits_a^b f(x)g(x)\, dx = f(c) \int\limits_a^b g(x)\, dx$.
(This result is called the generalized mean value theorem of Riemann integrals.)

64. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $g(x) = \int\limits_0^x (x - t)f(t)\, dt$ for all $x \in \mathbb{R}$. Show that $g''(x) = f(x)$ for all $x \in \mathbb{R}$.

65. Let $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{if } 1 < x \le 2, \end{cases}$ and let $F(x) = \int_0^x f(t)\, dt$ for all $x \in [0, 2]$.
Is $F : [0, 2] \to \mathbb{R}$ differentiable? Justify.

66. If $f : [0, 1] \to [0, 1]$ is continuous, then show that the equation $2x - \int_0^x f(t)\, dt = 1$ has exactly one root in $[0, 1]$.

67. Examine whether the following improper integrals are convergent.
(a) $\int_0^\infty e^{-t^2}\, dt$
(b) $\int_{-\infty}^\infty te^{-t^2}\, dt$
(c) $\int_0^1 \frac{dt}{\sqrt{t-t^2}}$

68. Determine all real values of p for which the integral $\int_1^\infty t^p e^{-t}\, dt$ converges.

69. Find the area of the region enclosed by the curve $y = \sqrt{|x+1|}$ and the line $5y = x + 7$.

70. The region bounded by the parabola $y = x^2 + 1$ and the line $y = x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

71. The region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ (where $a > 0$) is revolved about the x-axis to generate a solid. Find the volume of the solid.

72. Find the area of the region that is inside the circle $r = 2\cos\theta$ and outside the cardioid $r = 2(1 - \cos\theta)$.

73. Find the area of the region which is inside both the cardioids $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$, where $a > 0$.

74. Consider the funnel formed by revolving the curve $y = \frac{1}{x}$ about the x-axis, between $x = 1$ and $x = a$, where $a > 1$. If V_a and S_a denote respectively the volume and the surface area of the funnel, then show that $\lim_{a\to\infty} V_a = \pi$ and $\lim_{a\to\infty} S_a = \infty$.