November 7, 2017.

Project title: Heisenberg uniqueness pairs on certain Lie groups and related problems.

Plan and objectives: In this project we shall be investigating a few problems of the following nature.

1. Let μ be a finite Borel measure which is absolutely continuous with respect to the arc length measure on a curve Γ and Λ be a non-empty set in $\mathbb{R}^2$. Then the pair (Γ, Λ) is called a Heisenberg uniqueness pairs (HUP in short) for μ if its Fourier transform $\hat{\mu}$ vanishes on Λ, implies $\mu = 0$.

   In an fundamental paper (**Ann. Math. (2) 173, (2011)**)[5], Hedenmalm and Montes-Rodriguez had investigated certain pairs in connection with HUP. Following is their main result. Suppose μ is supported on hyperbola $\Gamma = \{(x, y) : \ xy = 1\}$ and $\hat{\mu}$ vanishes on the lattice-cross $\Lambda = \alpha\mathbb{Z} \times \{0\} \cup \{0\} \times \beta\mathbb{Z}$. Then $\mu = 0$ if and only if $\alpha\beta \leq 1$, where $\alpha, \beta \in \mathbb{R}_+$. As a dual problem, they had constructed a weak* dense subspace of $L^\infty(\mathbb{R})$ which in turn solves the **one dimensional Kein-Gordon equation.** Further, they worked out a complete characterization of the Heisenberg uniqueness pairs corresponding to any two parallel lines.

   Thereafter, a considerable amount of work has been done. We are working on this problem and several articles are in progress. Now we briefly mention some of them.

   Heisenberg uniqueness pairs corresponding to the circle had been independently investigated by N. Lev [8] and P. Sjolin [12] in 2011. In 2012, F. J. Gonzalez Vieli [15] had generalized the above result for the sphere in $\mathbb{R}^n$.

   In 2013, Per Sjolin [13] had investigated some of the HUP corresponding to the parabola. In 2013, D. Blasi Babot [2] had given a characterization of HUP corresponding to the certain system of three parallel lines in the plane. However, the analogous problem for the finitely many parallel lines is still open. In 2014, P. Jaming and K. Kellay [6], had given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to the hyperbola, polygon, ellipse and graph of the functions $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$.

   In a recent article "Heisenberg uniqueness pairs for some algebraic curves in the plane" which appeared in **Adv. Math. 310 (2017)** we have investigated HUPs corresponding to the spiral, hyperbola, circle and the exponential curves using basic Fourier analysis. This article is available at http://arxiv.org/1506.07425.

   Further in this article, we have proved a characterization of the HUPs corresponding to the four parallel lines. In the latter case, we observe a phenomenon of interlacing of three trigonometric polynomials.

   Remarks and open problems observed in our article:

   (a) In this article, we have observed a phenomenon of interlacing of three totally disconnected dispensable disjoint sets $\Pi^{(3-j)^*}(\Lambda) : \ j = 0, 1, 2$ which are essentially derived out of the zero sets of four trigonometric polynomials.

(b) If the measure in question is supported on an arbitrary number of parallel lines, then the size of the dispensable sets would be larger. Indeed, the method used for the proof of result for four parallel lines would be highly implicit for a large number of parallel lines. However, since the dispensable sets are totally disconnected and are laying in the zero sets of finitely many trigonometric polynomials, then it would be an interesting question to analyze HUP corresponding to the finite number of parallel lines in terms of the **lower bound for Hausdorff dimension of the dispensable sets**.

(c) On the other hand, if we consider **countably many parallel lines**, then whether the projection $\Pi(\Lambda)$ would be still relevant after deleting the countably many dispensable sets having total Lebesgue measure zero seems to be a reasonable question. We leave these questions open for the time being.

2. We are trying to get analogous results in connection to Heisenberg uniqueness pairs for deferent setups, namely the Heisenberg group, the Euclidean motion groups, step two nilpotent Lie group and real hyperbolic spaces etc.

   (a) *A Non-harmonic cones are Heisenberg uniqueness pairs for the Fourier transform on* $\mathbb{R}^n$, R. K. Srivastava, (submitted).

   In this article, we prove that a cone is a Heisenberg uniqueness pair corresponding to the sphere as long as the cone does not completely lay on the level surface of any homogeneous harmonic polynomial on $\mathbb{R}^n$. DOI: arXiv:1507.02624

   (b) *Heisenberg uniqueness pairs for the Fourier transform on the Heisenberg group,* R. K. Srivastava (preprint).

   (c) *Non-harmonic cones are Heisenberg uniqueness pairs for the modified Fourier transform on the Heisenberg group,* R. K. Srivastava, (preprint).

   (d) *Heisenberg uniqueness pairs for the spectral projections on the Heisenberg group,* R. K. Srivastava, (preprint).

3. In addition, the problem of determining the Heisenberg uniqueness pair for the Fourier transform for a class of finite measures has also a significant similarity with the celebrated result due to M. Benedicks [1]. That is, support of a function in $L^1(\mathbb{R}^n)$ and support of its Fourier transform both can not have finite measure simultaneously. Later, a series of analogous problems to the Beniticks theorem have been investigated in various set ups, including the Heisenberg group and the Euclidean motion groups etc (see [9, 10, 14]).

   We are working on the analogous problems to Benedicks Theorem in the Euclidean motion groups, Heisenberg group motion group, step two nilpotent Lie group. Following are preprints of our on going work in connection to Benedicks Theorem.

   (a) D.K. Giri and R.K. Srivastava, Heisenberg uniqueness pairs for some algebraic curves and surfaces, **Trans. Amer. Math. Soc. (accepted), 2017.** DOI: arXiv:1605.06724

   (b) A. Chattopadhyay, D.K. Giri and R.K. Srivastava, Uniqueness of the Fourier transform on certain Lie groups, (submitted). DOI: arXiv:1607.03832

(c) A. Chattopadhyay, D.K. Giri and R.K. Srivastava, Uniqueness of the Fourier transform on the Euclidean motion group, (submitted). DOI: arXiv:1608.00825

4. We are considering to work for a complete characterization of eigenfunctions of Laplacian on Euclidean motion groups. This problem is still open. We are studying this this problem via deeper group structure and geometry of the space.

5. In an interesting article **(Ann. Math. (2) 170, (2009)** [7] R. Rowan and B. Simon has investigated the necessary and sufficient conditions for a positive measure to be the spectral measure of a half-line Schrodinger operator with square integrable potential in $\mathbb{R}^2$. We will try a variance of this result in higher dimensions as well well as different setups, namely Heisenberg group, the Euclidean motion groups, step two nilpotent Lie group, real hyperbolic spaces etc.

6. In a fundamental article **(Ann. Math. (2) 176 (2012)**[4], Rupert L. Frank and Elliott H. Lieb had investigated the sharp constants for the inequalities on the Heisenberg group whose analogues on Euclidean space $\mathbb{R}^n$ are the well known Hardy-Littlewood-Sobolev inequalities. We will study this problem via deeper group structure and geometry of the space on real hyperbolic spaces.

   Methodology: We use various tools and techniques developed for study of problems in Fourier Analysis (or Harmonic Analysis). Namely, spectral decomposition of self-adjoint unbounded operators, spherical harmonic decompositions (i.e. the generalized Fourier series on the unit sphere), spectral synthesis, method of convolution equations, PDE etc.

   Impact of this study to Mathematics and applications: This project will have an impressive impact on the development of the basic mathematical analysis, namely Fourier Analysis, Functional Analysis, PDEs, Theoretical Physics etc.

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