Spectral Theory:

Spectral theory is a way to characterize operators on Banach/Hilbert spaces, in terms of scalars. In finite dim space, these scalars are finitely many while in the infinite dim. spaces, these scalars could be a countable set (compact operators) or even the continuum.

This is one of the way, why does "spectral theory" is immense important while deciding the core behavior of a bounded linear operator.

To start with, we first consider the finite dim. case. Suppose

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n, \text{ and } T = (a_{ij})_{n\times n}$$

For $0 \neq x \in \mathbb{R}^n$, & some $\lambda \in \mathbb{R}$, write $Tx = \lambda x$.

Then

$$(*)\quad \left\{ \begin{array}{l} (T-\lambda I)x = 0. \text{ That is,} \\ T-\lambda I \text{ is not one-one} \\ \text{iff } T-\lambda I \text{ is not onto} \\ \text{iff } T-\lambda I \text{ is not invertible} \end{array} \right.$$

However, (*) does not continue to hold in the infinite dim. spaces, even no "point spectrum" exists at all. For example, the right

shift operator $R: l^2 \to l^2$ given by
$R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ is one-one but onto, with $\|R\| = 1$. Similarly, left shift operator, and hence both are not invertible.
This will give a sense as to why spectral theory is more important in case of infinite dim. spaces.
note that $T: l^2 \to l^2$ given by
$T(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots)$
is a compact operator having no eigen-value, which is not self adjoint.
Further, $T: L^2[0,1] \to L^2[0,1]$ given by
$(Tf)(t) = t f(t)$ is a self adjoint non-compact operator having no eigen-value.

Ex. $\Delta := \frac{d^2}{dt^2} : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ is a self adjoint operator, which is not compact, having no eigen-value. For this, consider
$\Delta f = \lambda f$. Then $-x^2 \hat{f}(x) = \lambda \hat{f}(x)$
(by taking the F.T. of both the sides)
i.e. $(x^2 + \lambda)\hat{f}(x) = 0 \Rightarrow \hat{f}(x) = 0$ if $\lambda \neq x^2$.
Since $\hat{f}$ is cont, $\hat{f} = 0 \Rightarrow f = 0$ a.e.
Thus, f is not an eigen-vector.
However, if Ω is a bounded domain in $\mathbb{R}$ (or $\mathbb{R}^n$), then Δ on $L^2(\Omega)$ has only

discrete eigen-values. (Known as Dirichlet BVP), a land mark result in PDE.

Let X be a complex Banach space, and $T \in B(X)$. The set

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$$

is known as spectrum of T.

We see later that $\sigma(T)$ is a compact set in $\mathbb{C}$ and $|\lambda| \leq \|T\|$ for all $\lambda \in \sigma(T)$.

Thus the set $\rho(T) = \mathbb{C} \setminus \sigma(T)$, known as resolvent of T, is an open set in $\mathbb{C}$, and hence it gives a window to know more about spectrum via complex analysis.

For $\lambda \in \rho(T)$, we write $R_T(\lambda) = R(\lambda) = (T - \lambda I)^{-1}$.

We come to know that $R(\lambda)$ is a Banach-valued analytic function on $\rho(T)$.

Notice that if $|\lambda| > \|T\|$, then $\|T/\lambda\| < 1$, and hence $I - \frac{T}{\lambda}$ is invertible. That is,

$T - \lambda I$ is invertible & hence

$\sigma(T)^c = \rho(T)$ is non-empty set with $\left(B_{\|T\|}(0)\right)^c \subset \rho(T)$.

For an open set $D \subset \mathbb{C}$, let us consider

$f : D \to X$. We say f is analytic at $z_0 \in D$ if $\exists\, r > 0$ and seq^n $(a_n) \in X$ s.t.

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n a_n, \ \forall\, z \in B_r(z_0),$$

and series in RHS converges absolutely.

We can prove a similar result as to Liouville's theorem.

<u>Theorem</u>: Let $f : \mathbb{C} \to X$ be an entire function and $\sup_{z \in \mathbb{C}} \|f(z)\| < \infty$. Then f is a constant function on $\mathbb{C}$.

Pf: Let $h \in X^*$, and write $g(z) := h(f(z))$. Then g is a bounded entire function on $\mathbb{C}$.

Consider $f(z) = \sum_{n=0}^{\infty} (z-z_0)^n a_n, \ z \in B_r(z_0)$.

Since the series converges abso. on X, and h is cont,

$$g(z) = \sum_{n=0}^{\infty} (z-z_0)^n h(a_n)$$

$$\Rightarrow |g(z)| \le \sum_{n=0}^{\infty} |z-z_0|^n \|h\| \, \|a_n\| < \infty.$$

Hence, by usual Liouville's theorem g is constant $\Rightarrow$ $g(z) = g(0) \Rightarrow h(f(z) - f(0)) = 0$, $\forall\, h \in X^*$. Since X^* separates points, it follows that $f(x) = f(0)$.

Note that if $\lambda, \mu \in \rho(T)$, then

$$R(\lambda) - R(\mu) = R(\lambda)(\lambda - \mu)I\, R(\mu)$$
$$= (\lambda - \mu) R(\lambda) R(\mu) \quad \text{---} (*)$$

$\Rightarrow R(\lambda) R(\mu) = R(\mu) R(\lambda)$. Also, it follows that $T - \lambda I = (T - \mu I)(I - (\lambda - \mu) R(\mu))$ —(**)

Now, let $\lambda_0 \in \rho(T)$ and λ be close to λ_0. Then $T - \lambda I = (T - \lambda_0 I)\{I - (\lambda - \lambda_0) R(\lambda_0)\}$

Recall that $I - (\lambda - \lambda_0) R(\lambda_0)$ is invertible if $|\lambda - \lambda_0| \, \|R(\lambda_0)\| < 1$. That is, if

$$|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0)\|} = \delta_0 \text{ (say)}.$$

Then $\lambda \in \rho(T)$. Thus, $\rho(T)$ is a non-empty open set in $\mathbb{C}$.

Further, by using commutativity of resolvent, we can write

$$R(\lambda) = (T - \lambda I)^{-1} = R(\lambda_0)\{I - (\lambda - \lambda_0) R(\lambda_0)\}^{-1}$$
$$= \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R(\lambda_0)^{n+1}.$$

This implies that the map $\lambda \longmapsto R(\lambda)$ is a analytic function on $\rho(T)$ into $B(X)$.

Moreover, $\rho(T)$ is open implies $\sigma(T)$ is closed. Since $\sigma(T) \subset B_{\|T\|}(0)$, it follows

$\sigma(T)$ is a compact set in $\mathbb{C}$.

Note that $\sigma(T) \neq \Phi$. If $\sigma(T) = \Phi$, then $\rho(T) = \mathbb{C}$ & $R(\lambda)$ being bounded entire function is constant by Liouville's thm, which is absurd, because

$$(B_{\|T\|}(0))^{c} \subset \rho(T).$$

(Note that $\|R(\lambda)\| \leq \sum_{k=0}^{\infty} \frac{\|T\|^{k}}{|\lambda|^{k+1}} = \frac{1}{|\lambda| - \|T\|} \to 0$ as $|\lambda| \to \infty$, hence $R(\lambda)$ is bounded).

Spectral radius:

for $T \in B(X)$, the spectral radius of T is the smallest disc in $\mathbb{C}$ that contains $\sigma(T)$. Hence

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

We know that for $\lambda \in \sigma(T)$, $|\lambda| \leq \|T\|$.

Hence $r(T) \leq \|T\|$. —— (*)

Equality in (*) need not hold as $r(T) = 0$ for nilpotent matrix.

(A) It is important to note that if $S, T \in B(X)$ and $ST = TS$. Then ST is invertible iff S & T both invertible.

Suppose ST is invertible, and $Tx = 0$.

If $x \neq 0$, then $ST(x) = 0$ for $x \neq 0$.
$\Rightarrow ST$ is not 1-1. Further if $T(X) \subsetneq X$, then $ST(X) \subsetneq S(X) \subseteq X$.

<u>Lemma</u>: Let X be a complex Banach space, and $T \in B(X)$. Then

$$\sigma(T^n) = \{\mu^n : \mu \in \sigma(T)\}$$

Pf: For $\lambda \in \mathbb{C}$,

$$t^n - \lambda = (t - \lambda_1) \cdots (t - \lambda_n) \quad \text{——— (1)}$$

$\Rightarrow T^n - \lambda I = (T - \lambda_1 I) \cdots (T - \lambda_n I)$ (Note that).
Hence $T^n - \lambda I$ is invertible iff each of $T - \lambda_j I$ is invertible. Hence $T^n - \lambda I$ is not invertible iff at least one of $(T - \lambda_j I)$ is not invertible.

$\Rightarrow \lambda \in \sigma(T^n)$ iff $\lambda_j \in \sigma(T)$ for some j.
Then from (1), $\lambda = \lambda_j^n$.
Thus, $\lambda \in \sigma(T^n)$ iff $\lambda = \mu^n$ for some $\mu \in \sigma(T)$.
Hence, $\sigma(T^n) = \{\mu^n : \mu \in \sigma(T)\}$

<u>Theorem (Gelfand)</u>:
Let X be a complex Banach space and $T \in B(X)$. Then spectral radius of T is given by $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$

proof: Since $\sigma(T^n) = \{\lambda^n : \lambda \in \sigma(T)\}$, by by taking supremum of both the sides we get $r(T^n) = (r(T))^n$.

$\Rightarrow (r(T))^n \leq \|T^n\| \Rightarrow r(T) \leq \|T^n\|^{1/n}$

i.e. $r(T) \leq \underline{\lim} \|T^n\|^{1/n}$ ——— (1)

Note that if $|\lambda| > r(T)$, then $\lambda \in \rho(T)$ and $R(\lambda) = T - \lambda I = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ conv. abso. in $B(X)$. Hence, $\frac{\|T^n\|}{|\lambda^{n+1}|} \leq C, \ \forall n \geq 1$.

$\Rightarrow \|T^n\|^{1/n} \leq C^{1/n} |\lambda|^{(1+\frac{1}{n})}$ ——— (2)

$\overline{\lim} \|T^n\|^{1/n} \leq |\lambda|, \ \forall |\lambda| > r(T)$

$r(T) \leq \underline{\lim} \|T^n\|^{1/n} \leq \overline{\lim} \|T^n\|^{1/n} \leq r(T)$.

That is, $r(T) = \lim \|T^n\|^{1/n}$.

Note that from (2), we can infer that

$r(T) = \lim \|T^n\|^{1/n} = \inf \|T^n\|^{1/n}$.

Ex. Let $T \in B(C^2[0,1])$ be defined by

$(Tf)(x) = \int_0^x f(t)\,dt$

Find the radius of conv. of T.

Remark: Spectral radius formula does not holds if T is $\pi/2$ rotation on $\mathbb{R}^2$.

First stage of decomposition of spectrum:

The operator $T-\lambda I$ could primarily be not invertible on X in three ways:

(i) $(T-\lambda I)$ is not one-one.
We denote all such λ by $\sigma_p(T)$, the set of point spectrum.
Since $T-\lambda I$ is not 1-1, $\exists$ $x \neq 0$ s.t.

$$(T-\lambda I)x = 0 \Rightarrow Tx = \lambda I x = \lambda x.$$

That is, λ is an eigen-value of T.
$\ker(T-\lambda I) =$ eigen-space of T corresponding to λ. (NVI, as for some cases, they are sufficient to diagonalize a given operator)

(ii) When $(T-\lambda I)$ is one-one and $(T-\lambda I)X$ is a <u>proper</u> dense subspace of X. Such spectrum is called continuous spectrum and we denote it by $\sigma_c(T)$.

Note that surjectivity is little more analytic than injectivity.

ex. $T: l^2 \to l^2$ by

$$T(x_1, x_2, \cdots) = (x_1, \frac{x_2}{2}, \cdots) \in l^2$$

& $\overline{T(l^2)} = l^2$ since $c_{00} \subset T(l^2) \subset l^2$, as

$(y_1, y_2, \cdots, y_n, 0, \cdots) \in c_{00} \Rightarrow (y_1, 2y_2, \cdots, ny_n, 0 \cdots) \in c_{00}$

$\Rightarrow (x_1, \frac{1}{2} 2y_2, \ldots \frac{1}{n} \cdot n y_n, 0, \ldots) \in \ell^2$ etc.

(iii) $(T-\lambda I)$ is one-one but $(T-\lambda I)X$ is not dense in X. Such spectrum we call residual spectrum and is denoted by $\sigma_r(T)$. Hence

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

There is another way to charaterize ~~an~~ invertible operators on a Banach space, in terms of bounded below and dense range subspace. This will give another way to decompose the spectrum of T.

Let us recall the following result.

<u>Theorem:</u> Let X & Y be two Banach spaces and $T \in B(X,Y)$. Then T is invertible (T^{-1} exists & bdd) iff $\overline{T(X)} = Y$ and

$$\|Tx\| \geq k\|x\| \text{ for some } k > 0, \ \forall x \in X.$$

Proof: If T^{-1} exists & bdd. Then T is onto, and $\overline{T(X)} = Y$. For $x \in X$, let $y = Tx$. Then $\|T^{-1}y\| \leq k\|y\| \Rightarrow \|x\| \leq k\|Tx\|$.

Conversely, suppose $\overline{T(X)} = Y$ & T is bounded below. Then T is 1-1, due to $\|Tx\| \geq k\|x\|$.

<u>T is onto:</u> Let $y \in Y$. Then due to $\overline{T(X)} = Y$, $\exists$ $Tx_n \to y$. Then $\{Tx_n\}$ is a C.S. and hence $\{x_n\}$ is C.S. Since X is complete, $x_n \to x \in X$, and hence $Tx_n \to Tx$. This shows that $T(X)$ is closed.

Thus, $T(X) = \overline{T(X)} = Y$. That is, T is a continuous bijection. Hence by IMT, T^{-1} is bdd.

Now, let $T \in B(X)$ & $\lambda \in \sigma(T)$. If $(T-\lambda I)X$ is <u>not</u> dense in X, then set of all such λ we denote by $\sigma_{com}(T)$, called compression spectrum.

If $(T-\lambda I)$ is not bounded below, then we say λ is an approximate spectrum and the set is denoted by $\sigma_{app}(T)$.

Note that if $T-\lambda I$ is not bounded below, then for each $n \in \mathbb{N}$, $\exists$ $x_n \in B_X$ s.t.

$$\|(T-\lambda I)x_n\| < \frac{1}{n} \to 0.$$

Hence $\lambda \in \sigma_{app}(T)$ iff $\exists$ $x_n \in B_X$ such that $(T-\lambda I)x_n \to 0$.

From the previous Theorem we can deduce that $\sigma(T) = \sigma_{app}(T) \cup \sigma_{com}(T)$.

following relation holds:

(i) $\sigma_r(T) = \sigma_{com} - \sigma_p$, because $\lambda \in RHS$ iff $\overline{(T-\lambda I)} \neq X$ & $T-\lambda I$ is one-one.

(ii) $\sigma_c = \sigma - (\sigma_{com} \cup \sigma_p)$, because for $\lambda \in RHS$, $\overline{(T-\lambda I)X} = X$ & $T-\lambda I$ is 1-1.

Now, we can list all spectrum for convenient.

(i) σ_p - $(T-\lambda I)$ is not one-one

(ii) σ_c - $(T-\lambda I)$ is one-one & $\overline{(T-\lambda I)X} = X$

(iii) σ_r - $(T-\lambda I)$ is one-one but $\overline{(T-\lambda I)X} \neq X$

(iv) σ_{com} - $(T-\lambda I)X$ is not dense in X

(v) σ_{app} - $T-\lambda I$ is not bounded below.

<u>Spectrum of adjoint operator</u>:

Here, we shall correlate the spectrum of T & T^*. For this we need to recall the following result.

<u>Theorem</u>: Let X & Y be two Banach spaces and $T \in B(X,Y)$. Then T is invertible iff T^* is invertible.

Proof: We know that $T: X \longrightarrow Y$ &
$T^*: Y^* \longrightarrow X^*$ is defined by
$T^*(g)(x) = g \circ T(x)$.
Suppose T^* is invertible. Then for $Tx = 0$,
we get $g(Tx) = 0$, $\forall\, g \in Y^*$.
If $x \neq 0$, then $\exists\, 0 \neq f \in X^*$ s.t. $f(x) = \|x\|$.
For this f, $\exists\, g \in Y^*$ s.t. $T^*(g) = f$
(since T^* is onto). Thus,
$0 = T^*(g)(x) = f(x) = \|x\| \Rightarrow x = 0$ is
absurd. Hence T is 1-1.
Now, we claim T is onto: Recall that
$\operatorname{Ker} T^* = (R(T))^{\perp}$, where for $M \subset Y$,
$M^{\perp} = \{g \in Y^* : g(M) = \{0\}\}$.
Given T^* is one-one, $(R(T))^{\perp} = \{0\}$, and
this holds iff $\overline{R(T)} = Y$ (By HBT)
(Note that this one the central result related to HBT).
Now, we claim that $R(T)$ is closed.
Let $Tx_n \to y \in Y$. Then $\{Tx_n\}$ is a C.s.
in Y. Let S^* be the inverse of T^*
(i.e. $T^*S^* = S^*T^* = I^*$).

$$\|x_n - x_m\| = \sup\{|f(x_n - x_m)| : f \in X^*, \|f\| = 1\}$$

$$= \sup\{|T^* S^*(f)(x_n - x_m) : f \in X^*, \|f\| = 1\}$$

$$= \sup\{ S^*(f)(T(x_n - x_m)) : f \in X^*, \|f\| = 1\}$$

$$\leq \|T(x_n - x_m)\| \sup\{|S^*(f)| : f \in X^*, \|f\| = 1\}$$

$$= \|S^*\| \, \|Tx_n - Tx_m\| \longrightarrow 0$$

This implies x_n is a C.S. in X & hence $x_n \to x \in X$. Then $Tx_n \to Tx = y$. Thus, $T(X)$ is closed in Y, and $T(X) = \overline{T(X)} = Y$.

$\Rightarrow T \in B(X, Y)$ & a bijection. By IMT, T is invertible.

Conversely if T is invertible. Then

$\ker T^* = R(T)^{\perp} = \{0\} \Rightarrow T^*$ is 1-1.

For $T^*(g) = f$, we get $g \circ T = f$, $g = f \circ T^{-1}$. Hence T^* is onto.

Thus T^* is a cont bijection & hence by IMT, T^* is invertible.

Therefore, $T - \lambda I$ is not invertible iff $T^* - \lambda I$ is not invertible.

Remark: The def^n of adjoint operator on a Banach space is slightly different than on a Hilbert space. Because of that

$$(T-\lambda I)^*(g) = g\circ(T-\lambda I) = (T^* - \lambda I)(g).$$

i.e. $(T-\lambda I)^* = T^* - \lambda I$.

However, on Hilbert space we have

$$(T-\lambda I)^* = T - \bar{\lambda} I.$$

<u>Theorem</u>: Let $T \in B(X)$. Then $\sigma_{com}(T) \subset \sigma_p(T^*)$ and $\sigma_p(T) \subset \sigma_{com}(T^*)$.

Pf: Let $\lambda \in \sigma_{com}(T)$, then $\overline{(T-\lambda I)X} \neq X$.

Consider $M = \overline{(T-\lambda I)X}$. Then $\exists\, 0 \neq f \in X^*$ such that $f(M) = \{0\}$. That is,

$$f((T-\lambda I)(x)) = 0, \quad \forall x \in X$$

$$(T^* - \lambda I)(f)(x) = 0 \quad \forall x \in X$$

$$(T^* - \lambda I)f = 0.$$

$\Rightarrow \lambda$ is an eigen-values of T^*.

$$\Rightarrow \quad \sigma_{com}(T) \subset \sigma_p(T^*).$$

(ii) Let $\lambda \in \sigma_p(T)$. Then $\exists\, 0 \neq x \in X$ s.t

$$f((T-\lambda I)x) = 0, \quad \forall f \in X^*.$$

$$\Rightarrow g(x) = 0, \quad \forall g \in R(T^* - \lambda I).$$

i.e. $F_x(g) = 0, \ \forall g \in R(T^* - \lambda I)$.

Hence $F_x(g) = 0,\ \forall\ g \in X^* = \overline{R(T^* - \lambda I)}$.

$\Rightarrow F_x = 0$. But $\|F_x\| = \|x\| \Rightarrow x = 0$.

Thus $R(T^* - \lambda I)$ is not dense in X^*.

Spectrum of Compact operators

The spectral theory of compact operators is quite simple and very much close to that of a matrix. In fact, the spectrum of a compact operator consists of only eigen-values union with $\{0\}$.

Theorem: Let X be a Banach space and T is a compact operator on X. Then for $\lambda \neq 0$, $N(T - \lambda I)$ is of finite dim & $R(T - \lambda I)$ is closed.

Since $\lambda \neq 0$, we can assume $\lambda = 1$.

Lemma: A finite dim or finite co-dim. closed subspace of a Banach space is complemented.

Pf: Let $M = \text{span}\{e_1, \dots, e_n\}$. Define

$\phi_i : M \to \mathbb{R}$ by $\phi_i(e_j) = \delta_{ij}$. Then by HBT ϕ_i can be extended to a conti linear functional on X.

Let $N = \bigcap_{j=1}^{n} N(\phi_j)$. Then $X = M \oplus N$.

If $x \in M \cap N$, then for $x = x_1 e_1 + \cdots + x_n e_n$,

$$0 = \phi_j(x) = x_j \delta_{jj}, \ \forall j \Rightarrow x = 0.$$

For $x \in X$, set $c_j = \phi_j(x)$, and write

$y = \sum_{j=1}^{n} c_j e_j$. Then $y \in M$. Now,

$$\phi_i(y) = \sum_{j=1}^{n} c_j \phi_i(e_j) \Rightarrow \phi_i(y) = \phi_i(x), \ \forall i.$$

Let $z = x - y$, then $\phi_i(z) = 0, \forall i \Rightarrow z \in N$.

Thus, $x = x - y + y = z + y$.

Similarly, if co-dim of M is finite.

<u>Proof of theorem:</u>

Given that X is Banach space & $T \in B_0(X)$.

Since $N(I-T)$ is a closed subspace,

$T : N(I-T) \longrightarrow X$ is a compact operator, because T is compact.

Notice that $T(N(I-T)) = I$, & T is cpt, unit ball in $N(I-T)$ is compact. Therefore, $\dim N(I-T) < \infty$.

Since $N(I-T)$ is closed & finite dim, it must be complemented. Say $X = N(I-T) \oplus M$, where M is closed.

Now, let $S = I - T/_{M}$. Then $\|Sx\| \geq k\|x\|$ for some $k > 0$. If not, then $\exists \ \|x_n\| = 1$,

where $x_n \in M$ s.t. $\|Sx_n\| < \frac{1}{n} \to 0$.
Since T is compact, w.l.g. we can assume that $Tx_n \to x_0$ (say). This implies

$$x_n = (I-T)x_n + Tx_n = Sx_n + Tx_n \to x_0.$$

$$\Rightarrow Sx_0 = 0 \Rightarrow x_0 \in N(I-T).$$

Also M is closed, & $x_n \to x_0 \Rightarrow x_0 \in M$.
Hence $x_0 = 0$, but $\|x_n\| = 1$, it follows that $\exists\, k > 0$ s.t. $\|Sx\| \geq k\|x\|$, $\forall x \in M$.
This will lead to us that $R(I-T)$ is closed.

As I told, spectral theory of cpt operator is simple, we are going to see that injectivity of cpt operator is equivalent to surjectivity (as in finite dim) upto a small perturbation.

<u>Theorem</u>: <u>(Fredholm Alternative)</u>:
Let X be a Banach space and $T \in B_0(X)$.
Then for $\lambda \neq 0$, $N(T-\lambda I) = \{0\}$ iff $R(T-\lambda I) = X$.

In other words, $Tx - \lambda x = y$ has solution for each $y \in X$ iff $Tx - \lambda x = 0$ has only solution $x = 0$.

Note that Fredholm alternative need not hold for $\lambda = 0$. For this, consider $X = C[0,1]$ and $Tf(x) = \int_0^x f(t)\,dt$, $f \in X$. Then T is compact, $N(T) = \{0\}$ and

$$R(T) = \{g \in C'[0,1] : g(0) = 0\}.$$

But $Tf = g$ has not solution for every $g \in X$.

<u>Lemma</u>: If $T \in B_0(X)$, then $R(I-T)$ has finite dimentional complement.

proof: We know that $\dim(\mathrm{Ker}(I-T)) < \infty$, and hence $\exists\ N \subset X$ s.t. $X = N \oplus M$, where $N = \mathrm{Ker}(I-T)$. Let $S = I-T$ and $N^k = \mathrm{Ker}(S^k)$, where $S^k = S^{k-1} \circ S$, $k = 1, 2, \dots$ and $S^0 = I$. Since $S^k = (I-T)^k = I - T_k$, for some $T_k \in B_0(X)$, $\dim N_k < \infty$, by previous theorem. Now, let $M_k = S^k(X) = S^k(M)$. Then

$$N_0 \subset N_1 \subset \cdots \text{ and } M_0 \supset M_1 \supset \cdots$$

We claim that $\exists\ n \;\&\; m$ s.t. $M_n = M_{n+1} = M_{n+2}$ and $N_m = N_{m+1}$. If not, then $\exists$ a seq^n $y_j \in M_j$ with $\|y_j\| = 1$ such that

$$\mathrm{dist}(y_j, M_{j+1}) > 1/2. \quad \text{(By Riesz lemma)}$$

For $m > n$, $Ty_m - Ty_n = (I-S)y_m - (I-S)y_n$

$Ty_m - Ty_n = y_m - Sy_m + Sy_n - y_n = z - y_n$.

Since $M_m \subset M_n$, $\text{dist}(y_n, M_m) > 1/2$.

Note that $z \in M_m$. Hence $\|Ty_m - Ty_n\| > 1/2$, which is impossible since T is compact.

Hence $M_m = M_{m+1} = M_{m+2} \cdots$

Similarly, we can find m s.t.

$$N_m = N_{m+1} = N_{m+2} \cdots$$

Now, let $p = \max\{m, n\}$. We claim that

$$X = N_p \oplus M_p.$$

Note that for $x \in X$, $S^p(x) \in M_p$ and

$$S^p(M_p) = S^p(S^p(X)) = S^{2p}(X) = S^p(X) = M_p.$$

That is, $S^p(x) \in M_p = S^p(M_p)$. Hence $\exists\, y \in M_p$ s.t. $S^p(x) = S^p(y) \Rightarrow x - y \in \text{Ker}\, S^p = N_p$.

Thus, $x = x - y + y$, and $X = N_p \oplus M_p$.

Therefore, $\text{Co-dim}(M_p) = \dim(N_p) < \infty$.

<u>Proof of Fredholm alternative:</u>

We need to prove that

$$\text{Ker}(I-T) = \{0\} \text{ iff } R(I-T) = X$$

$$\& \ \text{Ker}\, S = \{0\} \text{ iff } S(X) = X$$

$$\text{That is, } N_1 = \{0\} \text{ iff } M_1 = M_0 = X.$$

Suppose $N = \{0\}$, and S is not onto.

That is, $M_1 \neq M_0$. We know that $\exists\, p \in \mathbb{N} \cup \{0\}$ s.t. $M_m = M_p$, $\forall\, m \geq p$. Let m_0 be

smallest integer s.t. $M_{m_0-1} \neq M_{m_0} = M_{m_0+1}$.
choose $u \in M_{m_0-1} \setminus M_{m_0}$. Then $S(u) \in M_{m_0} = M_{m_0+1}$.
Hence $\exists\ v \in M_{m_0}$ such that $S(u) = S(v)$,
but $v \neq u$. That is, $S(u-v) = 0$ for $u-v \neq 0$,
which is a contradiction that $N = \{0\}$.

On the other hand suppose S is onto, and
$N_1 = S(X) \neq \{0\}$. Then $\exists\ 0 \neq x_1 \in N_1$.
Since S is onto, $\exists\ x_2 \in X$ s.t. $Sx_2 = x_1$.
Note that $x_2 \in N_2 \setminus N_1$ & $x_2 \neq 0$. Once again by surjectivity of S, $\exists\ x_3 \in X$ s.t.

$$Sx_3 = x_2, \quad x_2 \in N_2 \setminus N_1 \text{ etc.}$$

That is $Sx_{k+1} = x_k$ for some $x_k \in N_k \setminus N_{k-1}$.
But $\exists\ p$ s.t. $N_m = N_p,\ \forall\ m \geq p$.
This implies, $Sx_{p+1} = x_p$, $x_p \in N_p \setminus N_{p-1}$,
and $x_{p+1} \in N_{p+1} = N_p \Rightarrow S^p(x_{p+1}) = 0$.

$$\Rightarrow 0 = S^p(x_{p+1}) = S^{p-1}(x_p) = \cdots = x_1,$$

which is a contradiction. Thus $N_1 = \{0\}$.

Proposition: Let X be an infinite dimensional Banach space and $T \in B_0(X)$.
Then $\sigma(T) = \sigma_p(T) \cup \{0\}$.

Proof: Since T is cpt, T cannot be invertible,
hence $0 \in \sigma(T)$. Let $0 \neq \lambda$, then for $\lambda \notin \sigma_p(T)$,

operator $T-\lambda I$ is one-one, and hence by Fredholm alternative, $T-\lambda I$ is invertible. Thus, $\lambda \notin \sigma(T)$. Hence $\sigma(T) = \sigma_p(T) \cup \{0\}$.

<u>Theorem</u>: Let X be a Banach space, and $T \in B_0(X)$. Then $\sigma_p(T)$ is countable and has only one possible limit point 0.

Pf: It is enough to show that

$\sigma_{p,\lambda}(T) = \sigma_p(T) \cap \{\lambda : |\lambda| \geq \epsilon\}$ is finite.

If not, then let $x_n \in X$ be such that $\|x_n\| = 1$ and $Tx_n = \lambda_n x_n$. Note that $\{x_n\}$ is a L.I. set. Write

$$M_n = \text{span}\{x_1, \cdots x_n\}, \quad n \geq 1.$$

Now, for $n > 1$, $\exists\ y_n \in M_{n-1}$, such that

$$\text{dist}(y_n, M_{n-1}) > 1/2 \quad \text{(by Riesz lemma)}.$$

Here, $y_n = \alpha_1 x_1 + \cdots + \alpha_n x_n$, and hence

$$Ty_n = \alpha_1 \lambda_1 x_1 + \cdots + \alpha_n \lambda_n x_n.$$

$\Rightarrow Ty_n - \lambda_n y_n \in M_{n-1}$.

For $n > m$,

$$Ty_n - Ty_m = \lambda_n y_n + \underbrace{Ty_n - \lambda_n y_n}_{M_{n-1}} - \underbrace{Ty_m}_{M_m \subset M_{n-1}}.$$

$\Rightarrow Ty_n - Ty_m = \lambda_n y_n + z$, for some $z \in M_{n-1}$.

$\Rightarrow \|Ty_n - Ty_m\| = |\lambda_n| \, \|y_n + \frac{z}{\lambda_n}\| > \frac{1}{2}\epsilon$,

which is a contradiction. $\sigma_p(T) \subseteq \bigcup_{n=1}^{\infty} \sigma_{p,1/n}(T)$.

Ex. Let $T : L^2[0,1] \longrightarrow L^2[0,1]$ be given by

$$T(f)(t) = tf(t).$$

Show that $\sigma_p(T) = \phi$, and

$$\sigma(T) = \sigma_{app}(T) = \sigma_c(T) = [0,1].$$

Notice that $\|T\| \le 1$ and hence $\sigma(T) \subset [-1,1]$.

(i) If for $\lambda \in \mathbb{C}$, & $f \in L^2[0,1]$, be such that $Tf = \lambda f$, then $(t-\lambda) f = 0 \Rightarrow f = 0$ a.e. Hence $\sigma_p(T) = \Phi$.

(ii) (a) $[0,1] \subset \sigma_{app}(T)$.

Let $\lambda \in [0,1]$. Then $\exists\ \epsilon > 0$ s.t. $[\lambda, \lambda+\epsilon] \subset [0,1]$ or $[\lambda-\epsilon, \epsilon] \subset [0,1]$.

Consider $[\lambda, \lambda+\epsilon] \subset [0,1]$, and define

$$f_\epsilon = \frac{1}{\sqrt{\epsilon}} \chi_{[\lambda, \lambda+\epsilon]}$$

Then $\int_0^1 |f_\epsilon|^2 dt = \int_\lambda^{\lambda+\epsilon} \frac{1}{\epsilon} dt = 1.$

$\Rightarrow f_\epsilon \in L^2[0,1]$, and

$$\|(\lambda I - T) f_\epsilon\|^2 = \int_\lambda^{\lambda+\epsilon} \frac{1}{\epsilon} (\lambda - t)^2 dt = \frac{\epsilon^2}{3} \longrightarrow 0$$

as $\epsilon \to 0$. Hence $[0,1] \subset \sigma_{app}(T)$.

(b) Let $\lambda \in \mathbb{C} \setminus [0,1]$. Then for $f \in L^2[0,1]$,

$$g(t) = \frac{f(t)}{\lambda - t} \in L^2[0,1].$$ This implies

$(\lambda I - T) g(t) = f(t)$. That is, $\lambda I - T$ is onto, and $\sigma_p(T) = \phi \Rightarrow \lambda I - T$ is invertible.

$\Rightarrow \sigma(T) \subseteq [0,1] \subseteq \sigma_{app}(T)$.

Hence $\sigma_{app} = \sigma(T) = [0,1]$.

(c) for $\lambda \in [0,1]$, $R(\lambda I - T)$ is dense in $L^2[0,1]$.

For $f \in L^2[0,1]$, write

$$f_n(t) = \begin{cases} f(t) & \text{if } |t-\lambda| \geq 1/n \\ 0 & \text{if } |t-\lambda| < \frac{1}{n}. \end{cases}$$

Then $\displaystyle\int_0^1 |f_n - f|^2 = \int_{|t-\lambda|<\frac{1}{n}} |f|^2 dt = \int_{|s|<\frac{1}{n}} |f(\lambda+s)|^2 ds$, $s = t-\lambda$.

$$= \int_{|s|<\frac{1}{n}} |\tilde{f}(s)|^2 ds \to 0$$

(by absolute cont. of functions in $L^1[0,1]$).

$\Rightarrow \|f_n - f\|_2 \to 0$.

Write $g_n(t) = \dfrac{f_n(t)}{\lambda - t}$. Then $g_n \in L^2[0,1]$

and satisfies $(\lambda I - T)g_n = f_n \xrightarrow{L^2} f$.

Notice that $R(\lambda I - T)$ is a proper dense subspace of $L^2[0,1]$ and $\sigma_p(T) = \Phi$. Hence, $\lambda I - T$ is 1-1. Thus, $[0,1] \subseteq \sigma_c(T)$.

Hence $\sigma_{app}(T) = \sigma_c(T) = \sigma(T) = [0,1]$.

Finally, we can see that $L^2[0,\lambda] \subset L^2[0,1]$ is T-invariant, which we need later.