Chapter 5  Gyroscope

Gyroscope is a spatial mechanism as shown in Figure 1 and generally employed for the control of angular motion of a body.

![Diagram of Gyroscope]

If we attempt to move some of its parts, it does not only resist this motion but even evades it. This resistance to change in the direction of rotational axis is called the **gyroscopic effect**.

Applications of gyroscopes are:

1. For directional control e.g.
   (i) gyro compass : for air planes & ships.
   (ii) inertial guidance control system: for missiles & space travel.

2. Gyroscopic effects are encountered in the bearings of
   (i). an automobile when it makes a turn and
   (ii) a jet engine shaft as the airplane changes direction.

3. For high speed rotors precession (see Figure 2) becomes more and more predominant and should be accounted in the design process.

![Diagram of Precessional Motion]

Figure 1. Gyroscope

Figure 2. Precessional motion of the disc
Gyroscopic Couple:
It can be easily studied using the principle of angular momentum. Angular velocity is a vector quantity. Change in magnitude and direction of angular velocity results in angular acceleration: Let in Figure 3, \(OA\) and \(OB\) are in \(x-z\) plane, \(\Delta \theta\) is the angular displacement of \(OA\) and \(OC\) is the angular displacement vector. Similarly angular velocity, angular acceleration and angular momentum are also vector quantity.

\[
\begin{align*}
Y \\
C & \Delta \theta \ \vec{\omega} \\
X & \\
O & \\
A & \\
Z \\
\end{align*}
\]

![Figure 3. Angular displacement vector](image)

**Linear momentum:** It is defined as

Linear momentum \(=\) mass \(\times\) velocity \(= mv\) \(\text{(1)}\)

The direction and sense of the linear momentum are same as linear velocity.

![Figure 4. A particle in motion (Linear momentum)](image)

**Angular momentum:** It is defined as the moment of linear momentum.

Angular momentum \(= H = (mv)r = (mr^2)\omega = I\omega\) \(\text{(2)}\)

where \(I\) is the mass of inertia about it’s axis of rotation and \(\omega\) is the angular velocity.

![Figure 5. Angular momentum](image)
Direction of angular momentum will be same as angular velocity.

\[ H = m v k = m \omega k k = m k^2 \omega = I \omega \] where \( I = m k^2 \) \hfill (3)

The gyroscopic couple is given as \( C = I \omega_p \omega_s \), where \( \omega_p \omega_s \) is the acceleration components. The direction of torque vector will be along the spin vector on rotating spin vector by 90° in the direction of precession vector (as shown in Figure 6).

Figure 6. Gyroscopic Torque

To cause precession of a spinning body, an external torque must be applied to the body (rotor) in a plane normal to the plane in which the spin axis is precessing. Our objective is to determine the required external torque to produce precession motion as specified or vice versa.

**Gyroscopic action on aeroplane structure:** Let \( C \) be the couple on the rotor by the plane or external couple the \( -C \) will be the reaction of the plane on the rotor.

Figure 7. Motion of an aeroplane

Figure 8. Gyroscopic couple on the aeroplane
**Exercise:** The rotor of a turbojet engine has a mass 200 kg and a radius of gyration 25 cm. The engine rotates at a speed of 10,000 rpm in the clockwise direction if viewed from the front of the aeroplane. The plane while flying at 1000 km/hr. turns with a radius of 2 km to the right. Compute the gyroscopic moment the rotor exerts on the plane structure. Also, determine whether the nose of the plane tends to rise or fall when the plane turns.

(Note: Here due to reaction couple of gyroscopic effect aeroplane will pitch about transverse axis which in turn give rise to a gyroscopic couple which will try to rotate the aeroplane in opposite to the direction of rotation. The rate of change of angular momentum \( \frac{dH}{dt} = I \frac{d\omega}{dt} = I \alpha \). By Newton’s Law: \( \frac{dH}{dt} = T = \) Torque to produce the angular acceleration, hence \( T = I \alpha \).

- **A rotor mounted on two bearings:** A rotor is spinning with constant angular velocity \( \omega_s \), the angular momentum is given by \( H = I \omega_s \). Let S-P-G are rectangular coordinate system (see Figure 9).

![Figure 9. A rotor mounted on two bearings](image-url)
\( OS \) is the spin axis, \( OP \) is the precession axis, \( OG \) is the gyroscopic torque axis and \( F \) is the action force on the shaft so that \((-F)\) is the reaction force on bearings. Let the shaft (or spin axis) presses through an angle \( \Delta \theta \) about \( P \) axis. Angular momentum will change from \( H \) to \( H' \), i.e.

\[
\overrightarrow{H'} = \overrightarrow{H} + \Delta \overrightarrow{H}
\]

where \( \Delta H \) is the change in angular momentum (due to change in direction of \( H \)). From \( \Delta OCD \)

\[
CD = (OC)(\Delta \theta) \quad \text{or} \quad \Delta H = H(\Delta \theta) \quad \text{or} \quad \Delta H = I\omega_s \Delta \theta
\]

Rate of change of angular momentum, \( \frac{dH}{dt} = \lim_{\Delta t \to 0} \frac{I\omega_s \Delta \theta / \Delta t = I\omega_s \omega_p}{\text{where } \omega_p \text{ is the uniform angular velocity of precession, hence}} \]

\[
C = I\omega_s \omega_p \quad (5)
\]

where \( C \) is the gyroscopic couple. The gyroscopic couple will have same sense and direction as \( \Delta H \) i.e. \( OC \). From right hand screw rule we will get the direction of torque i.e. clockwise about axis \( OG \) when seen from above. This is active couple acting on the disc. Whenever an axis of rotation or spin axis changes its direction a gyroscopic couple will act about the third axis. A reactive gyroscopic couple will be experienced by bearings through the shaft.

**Alternative deviation of gyroscopic couple**

![Figure 10. Gyroscopic couple on a rotating disc.](image-url)
Let $XX$ id the spin axis ($\omega_s$) and $YY$ is the precession axis ($\omega_p$). Particle $P$ has coordinates $(r, \theta)$ and mass of $dm$. The velocity $\omega_s r \perp OP$. Velocity component in: $ZZ$ direction = $\omega_s r \sin \theta = \omega_s y$ and in the $YY$ direction = $\omega_s r \cos \theta = \omega_s z$. Particle $P$ is having motion along the $z$ axis (or $\parallel$ to $z$-axis). Simultaneously, it is rotating about axis $Y-Y$. So a Coriolis component of acceleration = $2\omega_s y \omega_p$ acts $\perp$ to plane of paper and outwards as shown in the side view (Figure 10). Similarly for particle $P'$, the Coriolis acceleration component will be $2\omega_s y \omega_p$ and it acts $\perp$ to plane of paper and inwards as shown in the side view (Figure 10). The accelerating forces arising out of these Coriolis acceleration components, produce a couple $C$ about $ZZ$-axis.

Force due to acceleration of the particle $P = dm2\omega_s y \omega_p$

Moment about $ZZ$-axis = $dm2\omega_s y^2 \omega_p$, hence total moment about $ZZ$-axis

$$C = \int 2\omega_s \omega_p y^2 \, dm = 2\omega_s \omega_p I_{zz} = 2\omega_s \omega_p (1/2) I$$

where $I_{zz} = \int y^2 \, dm = (1/2) I$, where $I$ is the polar moment of inertia (for thin disc). $C$ acts along the $ZZ$-axis towards right. Moment about $YY$-axis = $dm2\omega_s z y \omega_p$, hence

Total moment about $YY$-axis = $\int 2\omega_s \omega_p z y \, dm = 2\omega_s \omega_p I_{zy}$ where $I_{zy} = \int z y \, dm = 0$

Also there is no Coriolis component of acceleration when we analyze the motion of particle in $y$-direction since $\omega_{zz} = 0$. If disc is not symmetric then $I_{zy} \neq 0$, so we will get $C_{zz}$ & $C_{yy}$ both.

Figure 11. Free body diagram of the disc
Thin rod rotating about its centroidal axis

In Figure 12, we have XX as the axis of spin and YY as the axis of precession. Due to Coriolis component of acceleration the force at point $P$, of the mass $dm$ is given as

$$dF = dm(2\omega_p \omega_s r \sin \theta)$$  \hspace{1cm} (1)

which is perpendicular to plane of the paper as shown in side view (Figure 12). Moment of this force about BB axis

$$dC_{BB} = dFr = dm(2\omega_p \omega_s r^2 \sin \theta)$$ \hspace{1cm} (2)

From parallel axis theorem, we have:

$$I_{BB} = I_{BB} + I_{AA} \approx 0$$ for thin rod, we have

$$C_{BB} = \int (2\omega_p \omega_s r^2 \sin \theta) \, dm = 2\omega_p \omega_s \sin \theta \int r^2 \, dm = 2\omega_p \omega_s \sin \theta I_{BB}$$  \hspace{1cm} (3)
\[ I = I_{BB} + I_{AA} \approx I_{BB} \]  

Equation (5) reduces to

\[ C_{BB} = 2I \omega_s \omega_p \sin \theta \]

which is \( \perp BB \), as shown in Figure 12 and can be obtained by right hand rule. Taking component of couple \( C_{BB} \) about \( yy \) and \( zz \) axis:

\[ T_{zz} = 2I \omega_s \omega_p \sin^2 \theta = I \omega_s \omega_p (1 - \cos 2\theta) \] \hspace{1cm} (7)

and

\[ T_{yy} = 2I \omega_s \omega_p \sin \theta \cos \theta = I \omega_s \omega_p \sin 2\theta \] \hspace{1cm} (8)

There are two gyroscopic couples respectively about \( zz \)-axis and \( yy \)-axis. This comes because of asymmetric body of revolution. \( I_{BB} \neq I_{AA} \). \( T_{zz} \) and \( T_{yy} \) are varying with \( \theta \), i.e. \( T_{zz} \) varies from 0 to \( 2I \omega_s \omega_p \) and \( T_{yy} \) varies from \( -I \omega_s \omega_p \) to \( I \omega_s \omega_p \). The above analysis is applicable to two bladed propeller or airscrew (Figure 13).

The above analysis can be extended to a multi-bladed airscrew. Let \( n \) be the number of blades \((n \geq 3)\), \( \alpha = \frac{2\pi}{n} \) is equally spaced angle between two blades.

\[ C_{zz} = I \omega_s \omega_p \quad \text{and} \quad C_{yy} = 0 \]

where \( I = nI_1 \), \( I_1 \) is the mass moment of inertia of each blade about an axis \( BB \). Let the moment of inertia of each blade about axis \( BB \) (i.e. \( \perp \) to blade) be equal to \( I_{BB} \) which in turn is equal to \( I_1 \), \( (I_1 = I_{BB} + I_{AA} \approx I_{BB}) \). Total moment of inertia of the airscrew about the axis of rotation is: \( I = nI_1 \).

Let us consider one of the blades, which is inclined to an angle \( \theta \) with \( zz \)-axis.
Figure 15. One of the blade position

Total moment about zz-axis

\[ C_{zz} = 2\omega_s \omega_p \int y^2 \, dm = 2\omega_s \omega_p I_1 \sin^2 \theta = I_1 \omega_s \omega_p (1 - \cos 2\theta) \tag{9} \]

Location of other blades is given by phase angle \( (n - 1)\alpha \). Summing up the moments due to all \( n \) blades, we have

\[
T_{zz} = I_1 \omega_s \omega_p \left[ 1 - \cos 2\theta \right] + I_1 \omega_s \omega_p \left[ 1 - \cos 2(\theta + \alpha) \right] \\
+ I_1 \omega_s \omega_p \left[ 1 - \cos 2(\theta + 2\alpha) \right] + \ldots + I_1 \omega_s \omega_p \left[ 1 - \cos 2\left( \theta + (n-1)\alpha \right) \right] \\
= I_1 \omega_s \omega_p \left[ n - \{ \cos 2\theta + \cos 2(\theta + \alpha) + \cos 2(\theta + 2\alpha) + \ldots + \cos 2\left( \theta + (n-1)\alpha \right) \} \right]
\]

which can be simplified as

\[
T_{zz} = I_1 \omega_s \omega_p \left[ n - \{ \cos 2\theta + \cos 2\left( \theta + (n-1)\alpha / 2 \right) \sin n\alpha / \sin \alpha \} \right]
\]

Since \( \alpha = 2\pi / n \)

For \( n > 2 \), \( \sin \alpha \neq 0 \) \( \rightarrow \) check for \( 2\pi / 3, \ 2\pi / 4 \ldots \)

\( \sin \alpha = 0 \) \( \rightarrow \) \( n\alpha = 2\pi \)

For all values of \( n > 2 \), \( T_{zz} = nI_1 \omega_s \omega_p \) \( \text{or} \) \( I_{zz} = I_1 \omega_s \omega_p \) \( \tag{10} \)

\( \sin n\alpha / \sin \alpha \) becomes \( \sin 2\pi / (\sin 2\pi / n) \), hence for

- \( n = 1 \), \( \sin 2\pi / \sin 2\pi = 1 \)
- \( n = 2 \), \( \sin 2\pi / \sin \pi = 0 \) \( \rightarrow \) 0
- \( n = 3 \), \( \sin 2\pi / (\sin 2\pi / n = 0 / \text{finite} \rightarrow 0) \)
- \( n = 4 \), \( \text{etc.} \) \( \rightarrow \) 0
Hence,

\[ T_{zz} = I_1 \omega_s \omega_p (1 - \cos 2\theta) \quad \text{for } n = 2, \quad \sin \alpha = 0, \quad \sin n\alpha = 0 \]

Moment about \( YY \)-axis for a blade, which makes angle \( \theta \) with \( ZZ \)-axis is:

\[ T_{yy} = I_1 \omega_s \omega_p \sin 2\theta \]

Total moment about \( YY \) for \( n \) blades is (phase angle \( (n-1)\alpha \))

\[ T_{yy} = I_1 \omega_s \omega_p \left[ \sin 2\theta + \sin 2(\theta + \alpha) + \ldots + \sin \left\{ 2 \left( \theta + (n-1)\alpha \right) \right\} \right] \]

(11)

\[ T_{yy} = 0 \quad \text{for } n > 2 \]

The sine series will be zero for all values \( n > 2 \). So with equations (10) and (11), we can conclude that for multi bladed screw with number of blades 3 and above is equivalent to a plane disc with polar mass moment of inertia \( I = nI \) about axis of rotation.

**Gyroscopic Stabilization**

Gyroscopes can be used for the stabilization of ships, aeroplanes etc. The motion can be termed as follows: Steering or yawing about vertical axis, pitching about transverse axis, and rotating about longitudinal axis. Generally rolling is more as compared to pitching or yawing. So gyroscope can be applied to reduce rolling. The basic purpose of gyroscope is to reduce the amplitude of the oscillations of the ship in a sea.

![Figure 16. Ship-motions](image)

**Basic principle of stabilization of gyroscope:**

The gyroscope shall be made to precess in such a way that the reaction couple exerted by the rotor (i.e. \(-C\)) shall oppose any disturbing couple (i.e. \( C_w \) external couple or applied couple to rotor)
which may act on the frame (of gyroscope), because of the waves of seas. If at every instant the reaction couple of the gyroscope (i.e. \(- C\)) and the applied or disturbing couple (\(C_w\)) are equal, then complete stabilization will be obtained. Due to change in slope of the wave and due to the buoyancy effect the ship will experience a couple. Generally waves are periodic (or sinusoidal). So ship will experience sinusoidal external couple in rolling. In order to maintain the ship stable, an equal and opposite reaction couple must be applied by the gyroscope.

![Figure 17. A ship in a rough sea](image)

In rolling, external couple is in transverse plane. So reaction couple from gyroscope should also act in same plane. (i.e. along longitudinal axis). So choice is there to choose spin axis either in the vertical or in transverse direction and accordingly for precession axis). This depends upon practical constraint. Precession is given to gyroscope manually (just like steering wheel). And it has to be given continuously. (or by variable speed motor with reversing direction capability). In Figure 18, we have \(C_w\) is the external couple applied by wave, \(oa\) I s the angular momentum, \(C_w\) is the \(\perp\) to plane of paper inward, and \(\Delta H\) is the Horizontal towards left. So precession should be such that it should produce a reaction couple, \(- C\), (i.e. same as magnitude and opposite to the direction or sense of the external couple, \(C\)) or gyroscope should experience an applied couple equal to, \(C(= C_w)\), such that reaction couple, \(- C\), will counterbalance external couple, \(C\). The external couple from wave will be in the same plane (i.e. \(C\) vector will be \(\perp\) to the plane of paper). But reaction couple, \(- C\), will change its direction (i.e. from \(\perp\) to the plane of paper as it precess due to \(\omega_p\)). So only component of reaction couple, \(- C\), will be available to counteract external couple, \(C\), of wave. Also slope of the wave \(\theta\) is also changing continuously. So \(C_w\) component will be maximum for maximum slope & zero when slope is zero.
Figure 18(a). Spin is axis along the longitudinal direction (Due to pitching no change of spin axis. So no gyroscopic action will be there. Yawning motion is rare. So this orientation is preferred).

Figure 18(b). Spin axis is along the vertical direction (Due to pitch change in spin axis. So gyroscopic action will be there).

Figure 19. Gyroscopic couple direction is ship

If \( T \sin \phi = C_W \) is disturbing couple applied by the wave, where \( \phi \) is the slope of the wave (see Figure 17). So, for \( \phi = 0, C_W = 0 \) and for \( \phi = \) maximum, \( C_W = T \sin \phi_{\text{max}} \). \( T_r = -C \) is the gyroscopic reaction couple that is applied by the gyroscope on the ship for an inclination of the plane of rotation by \( \theta \) to the horizontal, then

\[
T_r \cos \theta = T \sin \phi \quad \text{or} \quad I \omega_3 \omega_p \cos \theta = T \sin \phi
\]
When, \( \phi = \phi_{\text{max}} \), then \( \theta = 0 \) and when \( \phi = 0 \), then, \( \theta \to 90^\circ \) (impractical) end of quarter period \((t/4)\) of the wave. Therefore, we can reduce the amplitude of rolling if not fully prevent it. Reference: *The Automatic Stabilization of Ships*, By T.W. Chalmers.

**Stability of a four-wheel drive vehicle moving on a curved path**

Reaction at wheels:

(i) Due to weight, \( mg/4 \) where \( m \) is the mass of the wheel

(ii) Due to centrifugal force \( F_c = mv^2/R \), where \( v \) is the velocity of vehicle, \( R \) is the radius of the curved path, and \( m \) is the mass of vehicle. On taking moment balance on the transverse plane of the vehicle, we have \( F_c h = Ux \) where \( U \) is the reaction at the inner and outer wheels (having two wheels at each of the inner and outer sides) due to the centrifugal force. Hence, on each wheel the reaction force due to centrifugal force will be \( U/2 = F_c h/x \). Now total reaction \( = W/4 - U/2 \) at inner wheel and \( = W/4 + U/2 \) at outer wheel.

Figure 20.
(iii) Gyroscopic action due to four wheels: \[ T_w = 4I_w \omega_p \omega_p \], where \( I_w \) is the moment of inertia of each wheel and \( \omega_p = V/R \). The torque due to the engine, \( T_e = I_e \omega_e \omega_p \). Total torque is given by \( T_g = T_\omega \pm T_e \), where \( + \) sign when \( \omega_e \) and \( \omega_p \) are having same sense and \( - \) sign when \( \omega_e \) and \( \omega_p \) are having opposite sense. We have \( Vx = T_g \) which gives \( V = T_g / x \), where \( V \) is the reaction at the inner and outer wheels due to the gyroscopic couple.

Total reaction at each of inner wheel

\[ T_{inner} = W/4 - U/2 - V/2 \]
and outer wheel

\[ T_{outer} = W/4 + U/2 + V/2 \]

At high speed or in the sharp turn, \( T_{inner} \) may become zero, so for the stability of the vehicle,

\[ T_{inner} > 0 \quad \text{or} \quad W/4 > (U + V)/2 \]

**Example 1:** A trolley can with a total mass of 2700 kg runs on rail 1 m apart with a speed of 30-km/hr. The track is curved with a radius of 40 m towards the rigid of the driver. The car has four wheels each of diameter 70 cm and the total moment of inertia of each pair of wheels and the axle is 15 kg-m\(^2\). The car is driven by a motor running in the direction opposite to that of the wheels at a speed five times the speed of rotation of the wheels. The motor and the gear pinion have a moment of inertia 10 kg m\(^2\). The rails are at the same level and the height of the center of gravity of the car is 1 m above the rail level. Determine the vertical force exerted by each wheel on the rails.

**Solution:**

![Figure 21 Trolley in a curved path](image-url)
\[ m_j = 2700 \text{ kg}; \quad d = 1 \text{ m}; \quad d_w = 0.7 \text{ m}; \quad h = 1 \text{ m}; \quad V = 30 \text{ km/h} = 8.33 \text{ m/s}; \quad \omega_p = \frac{V}{R} = 0.2083 \text{ rad/s}. \]

For wheels: \( \pi d_w \omega_w = V; \quad \omega_w = 3.79 \text{ rad/s}. \)

For engine: \( \omega_k = 5\omega_w = 18095 \text{ rad/s} = 3.79 \text{ rad/s}. \)

Gyroscopic couple: \( C = (I_w \omega_w - I_k \omega_k) \omega_p = [2 \times 15 \times 3.79 - 10 \times 18.95] \times 0.2083 = -15.79 \text{ Nm}. \)

So the effect of net gyroscopic couple will to that outer wheel reactions will decrease (since direction of net gyroscopic couple is in the direction of \( C_k \)) and increase the reactions at inner wheels. Let due to gyroscopic effect the reaction at inner wheel is 2G. On taking moment about an axis passing through outer wheel we get.

\[ 2G_i \times \text{distance between tracks} = C \quad \text{or} \quad 2G_i \times 1 + (-15.79) = 0 \]

or \( G_i = 7.895 \text{ N} \) So \( G_i = 7.895 \text{ N} \) and \( G_0 = -7.895 \text{ N} \)

Due to weight: Force on each wheel \( w = \frac{mg}{4} = 6621.75 \text{ N} \)

Due to centrifugal force: Taking moment about the outer track, we have

\[ F_c \times 1 + (2F_{c_i}) \times 1 = 0; \quad \therefore \frac{mv^2}{l^2} \times 1 + (2F_{c_i}) \times 1 = 0 \]

or \( F_{c_i} = -2343.75 \text{ N} \) and \( F_{c_o} = +2343.75 \text{ N} \)

Vertical reactions at each of the outer wheel = 6621.75 - 7.895 + 2343.75 = 8997.6 \text{ N}

On each of the inner wheel: = 6621.75 + 7.895 - 2343.75 = 4285.9 \text{ N}.

**Example 2:** The wheels of a motorcycle have a moment on inertia 68 kg m$^2$ and engine parts a moment of inertia of 3.4 kgm$^2$. The axis of rotation of the engine crankshaft is parallel to that of the road wheels. If the gear ratio is 5 to 1, the diameter of the road wheels is 65 cm and the motor cycle
rounds a curve of 30.5 m radius at 60 km/hour, find the magnitude and direction of the gyroscopic couple.  

**Solution:**  

\[ I_{\text{wheel}} = 68 \, \text{kgm}^2; \quad I_{\text{Engine}} = 3.4 \, \text{kgm}^2; \quad n = \text{Gear ratio} = 5:1; \quad D_{\text{wheel}} = 65 \, \text{cm}, \quad R = 30.5 \, \text{m} \]

Velocity = 60 km/hr = 16.67 m/sec  

\[ \omega_{\text{motorcycle}} = \frac{16.67}{30.5} = 0.5464 \, \text{rad/sec} \quad \text{and} \quad \omega_{\text{wheel}} = \frac{16.67}{0.65/2} = 51.28 \, \text{rad/sec}. \]

\[ C_{\text{wheel axle}} = 68 \times 51.28 \times 0.5464 = 1905.31 \, \text{kgm}^2/\text{sec}^2 \quad \text{and} \quad C_{\text{engine}} = 3.4 \times 256.41 \times 0.5464 = 476.34 \, \text{kgm}^2/\text{sec}^2 \]

Note: \[ C = 1425.52 \, \text{kgm}^2 \]

Relative couple direction is on the fame of the frame of the motorcycle either from wheel axle or from engine seat.

**Exercise Problems:**

1. The rotor of a Jet airplane engine is supported by two bearings as shown in Figure 1. The rotor assembly including compressor, turbine, and shaft is 6672 N in weight and has a radius of gyration of 229 mm. Determine the maximum bearing force as the airplane undergoes a pullout on a 1830 m radius curve at constant airplane speed of 966 km/h and an engine rotor speed of 10,000 rpm. Include the effect of centrifugal force due to the pullout as well as the gyroscopic effect.