M/G/1 Queue

with

Vacations

**Vacation:** After a busy period, the server goes on vacation of random length. It examines the queue once again when it returns from the vacation.

-Multiple Vacations (possibly)-

If system still empty when the server returns from a vacation, it goes for another vacation. This continues until it finds system non-empty on return from vacation; it then resumes service normally.

-Single Vacation (per idle)-

After a busy period ends, server goes on only one vacation. If system is still empty when in returns, it stays and waits for a job to arrive.

Other models are also possible, i.e. server goes on (possibly multiple) vacations following the busy period until there are K waiting jobs.
Analysis of M/G/1 Queue with (Multiple) Vacations

Vacation Interval random (i.i.d) and independent of service times with

- Moments $\bar{V}$, $V^2$
- pdf $f_V(t)$
- cdf $F_V(t)$
- L.T. $L_V(s)$

Residual Life Approach

Imbedded Markov Chain Approach

Analysis using the Residual Life based Approach

$r(t)$: Residual Time for the Currently Ongoing Service or Vacation Time
X_i: i-th service time
V_j: j-th vacation time

Residual Time $r(\tau)$ - service time or vacation - for a M/G/1 Queue with (multiple) Vacations
Time Average of \( r(t) \) over \((0, t)\) is given by:

\[
\frac{1}{t} \int_0^t r(x) \, dx = \frac{1}{t} \sum_{i=1}^{M(t)} X_i^2 + \frac{1}{t} \sum_{j=1}^{L(t)} V_j^2
\]

where

\[
\begin{align*}
M(t) &= \text{Number of arrivals in the interval } (0,t) \\
L(t) &= \text{Number of vacation intervals in the interval } (0,t)
\end{align*}
\]

For \( t \to \infty \), the following limits are:

\[
\begin{align*}
R &= \lim_{t \to \infty} \frac{1}{t} \int_0^t r(x) \, dx \\
\lim_{t \to \infty} \frac{1}{M(t)} \sum_{i=1}^{M(t)} X_i^2 &= X^2 \\
\lim_{t \to \infty} \frac{1}{L(t)} \sum_{j=1}^{L(t)} V_j^2 &= V^2 \\
\lim_{t \to \infty} \frac{t(1-\rho)}{L(t)} &= \bar{V}
\end{align*}
\]

As for the basic M/G/1 queue considered earlier, this leads to

Mean Residual Time (service or vacation)

\[
R = \frac{1}{2} \lambda \overline{X}^2 + \frac{1}{2} (1-\rho) \frac{\overline{V}^2}{\overline{V}}
\]

Writing

\[
W_q = N_q \overline{X} + R = \lambda W_q \overline{X} + R
\]

gives

\[
W_q = \frac{\lambda \overline{X}^2}{2(1-\rho)} + \frac{\overline{V}^2}{2\overline{V}} \quad \rho = \lambda \overline{X}
\]

as the mean waiting time in queue seen by an arriving customer.

Knowing \( W_q \), the other parameters \( N_q, N, \) and \( W \) may be found.
Analysis using the Imbedded Markov Chain Approach

As for the basic M/G/1 queue, imbed Markov Chain of system states (denoting the number in the system) at the time instants \( t_i \) \( i=1, 2, 3, \ldots \), when the \( i \)th customer departs from the system

- \( \eta_i \) = Number of jobs left behind in the system by the \( i \)th departure
- \( a_i \) = Number of job arrivals during the \( i \)th service time

\[ A(z) = L_p(\lambda - \lambda z) \]

with \( A(1) = 1 \) \( A'(1) = \rho = \frac{\lambda}{\lambda - \lambda z} \) \( A''(1) = \lambda^2 \frac{\lambda - 1}{(\lambda - \lambda z)^2} \)

\( j \) = Number of jobs waiting for service when a busy period begins, \( j \geq 1 \)

\( f_j = P\{j \text{ customers starting the busy period}\} \quad j=1,2,\ldots,\infty \)

Generating Function for \( j \) (see Sec. 4.1.1 for details)

\[ F(z) = \sum_{j=1}^{\infty} f_j z^j = E\{z^j\} \]

\[ = \frac{L_V(\lambda - \lambda z) - L_V(\lambda)}{1 - L_V(\lambda)} \]

with \( F(1) = 1 \) \( F'(1) = \frac{\lambda V}{1 - L_V(\lambda)} \) \( F''(1) = \frac{\lambda^2 V^2}{1 - L_V(\lambda)} \)
Relating the state at the $i^{th}$ and $(i+1)^{th}$ instants, we get

$$n_{i+1} = a_{i+1} + j - 1 \quad \text{for} \quad n_i = 0$$

$$= n_i + a_{i+1} - 1 \quad \text{for} \quad n_i \geq 1$$

or

$$n_{i+1} = n_i + a_{i+1} - 1 + f[1 - U(n_i)]$$

(4.4)

$$P(z) = E\{z^{n_{i+1}} \} = E\{z^{n_i} \}E\{z^{i+1} \}$$

$$P(z) = A(z)E\{p_0 z^{i+1} + \sum_{n=1}^{\infty} z^{n-1} p_n \}$$

$$\Rightarrow \quad P(z) = p_0 A(z) \frac{1 - F(z)}{A(z) - z}$$

Evaluating $P(z) = p_0 A(z) \frac{1 - F(z)}{A(z) - z}$ at $z=1$, i.e. using $P(1)=1$, gives

$$p_0 = \frac{1 - p}{F'(1)}$$

(4.9)

and therefore

$$P(z) = (1 - p)\left(1 - L_v (\lambda - \lambda z) \right) \left( \frac{L_v (\lambda - \lambda z)}{\lambda - L_v (\lambda - \lambda z)} \right)$$

(4.10)

Note that though $P(z)$ was derived for the customer departure instants, it will also hold for the arrival instants and at an arbitrary time instant under equilibrium conditions.
• From $P(z)$, we can find the system state distribution either by inverting the generating function $P(z)$ or by expanding it in powers of $z$.

• The moments of the number in the system may be found directly using the moment generating properties of the generating function $P(z)$.

• Specifically, we get $N = P'(1) = \lambda \overline{X} + \frac{\lambda^2 \overline{X}^2}{2(1 - \lambda \overline{X})} + \frac{\lambda \overline{V}^2}{2\overline{V}}$.

• Knowing $N$, we can obtain $W$, $W_q$, and $N_q$ following our usual approach.

For example

$$W_q = \frac{\lambda \overline{X}^2}{2(1 - \rho)} + \frac{\overline{V}^2}{2\overline{V}}$$  \hspace{1cm} (4.11)$$

M/G/1 Queue with only one Vacation after Idle  \hspace{1cm} (Section 4.2)

$P(z) = p_0 \left( \frac{L_y (\lambda - \lambda z)}{z - L_y (\lambda - \lambda z)} \right) L_v (\lambda - \lambda z) - (1 - z) L_v (\lambda) - 1$

with $p_0 = \frac{1 - \lambda \overline{X}}{\lambda \overline{V} + L_v (\lambda)}$  \hspace{1cm} Using Imbedded Markov Chain

Directly using Residual Life Approach

$$W_q = \frac{\lambda \overline{X}^2}{2(1 - \lambda \overline{X})} + \frac{\overline{V}^2}{2 \left( \overline{V} + \frac{1}{\lambda} L_v (\lambda) \right)}$$
M/G/1 Queue with Exceptional First Service  (Section 4.3)

In this queue, the first customer starting a busy period requires a service time with a different distribution, i.e. $b^*(t)$ and $L_B^*(s)$ with moments $\bar{X}$ and $\bar{X}^2$.

An imbedded Markov Chain analysis will give

$$P(z) = p_0 \frac{L_B(\lambda - \lambda z) - zL_B(\lambda - \lambda z)}{L_B(\lambda - \lambda z) - \bar{z}}$$

(4.19)

with

$$p_0 = \frac{1 - \lambda \bar{X}}{1 - \lambda \bar{X} + \lambda \bar{X}^*}$$

(4.18)

The delay distribution for the FCFS case, may be found using

$$P(z) = L_B(\lambda - \lambda z).$$

This may then be used to find $W$ and $W_q$.

Alternatively, these may be found using a Residual Life Approach

$$W = \frac{\bar{X}}{1 - \lambda \bar{X} + \lambda \bar{X}^*} + \frac{\lambda \bar{X}^2}{2(1 - \lambda \bar{X})} + \frac{\lambda(\bar{X}^* - \bar{X}^2)}{2(1 - \lambda \bar{X} + \lambda \bar{X}^*)}$$

(4.21)

$$W_q = \frac{\lambda \bar{X}^2}{2(1 - \lambda \bar{X})} + \frac{\lambda(\bar{X}^* - \bar{X}^2)}{2(1 - \lambda \bar{X} + \lambda \bar{X}^*)}$$

(4.22)