

DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA-101 : Mathematics-I
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Tutorial Sheet No. 4
R. Alam, V. Wagh

Solution for Tutorial Sheet No.4

1. Let $f, g : A \rightarrow \mathbb{R}$ be uniformly continuous on A . Are the functions $f+g, fg, 1/f$ (provided $f(x) \neq 0$ for all $x \in A$) uniformly continuous on A ? What if A is a bounded subset of \mathbb{R} ? What if $A = [a, b]$?

Solution: That $f + g$ is U.C. on A is evident.

Next, fg is U.C. if both f and g are bounded (easy proof). To see what happens when both are not bounded, consider $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) := x$ and $g(x) := \sin(x)$. Then both f and g are U.C. but not fg . Indeed, consider $x_n := n\pi$ and $y_n := (n + 1/n)\pi$ for $n \in \mathbb{N}$. Set $h := fg$. Then $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ but

$$|h(x_n) - h(y_n)| = |y_n \sin(y_n)| = (n+1/n)\pi |\sin(\pi/n)| = \pi^2 \frac{|\sin(\pi/n)|}{\pi/n} + \pi/n |\sin(\pi/n)| \rightarrow \pi^2$$

as $n \rightarrow \infty$. This shows that h is not U.C.

If A is bounded then both f and g are bounded (Question 4 below). Hence fg is U.C.

On the other hand, if $|f(x)| \geq m > 0$, for some $m \in \mathbb{R}$, then $1/f$ is U.C. (easy proof). Now, consider $f : (0, 1) \rightarrow \mathbb{R}$ given by $f(x) := x$. Then f is U.C. but $1/f$ is not U.C. even though the domain of f is bounded.

Finally, if $A := [a, b]$ then $f + g, fg$ and $1/f$ are U.C. ■

2. Analyze the following functions for uniform continuity.

- (i) $f(x) := \sqrt{x^3}$, $x \in (2, 3)$; (ii) $f(x) := \sin(1/x)$, $x \in (0, 1]$;
(iii) $f(x) := 1/x^2$, $x \in (0, 1)$; (iv) $f(x) := x^3$, $x \in \mathbb{R}$; (v) $f(x) := 1/(1+x^2)$, $x \in \mathbb{R}$.

Solution: (i) Since limits of f exist as x tend to 2 and 3, by the continuous extension theorem, f is U.C. ■

(ii) It is easy to see that f is not U.C. Indeed, consider $x_n := 1/n\pi$ and $y_n := 1/(2n\pi + \pi/2)$. Then $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ but $\lim_{n \rightarrow \infty} (f(x_n) - f(y_n)) = 1$. ■

(iii) f is not U.C. Indeed, for the Cauchy sequence $(1/n)$, $n \geq 2$, the sequence $(f(1/n))$ is unbounded. ■

(iv) f is not U.C. Indeed, consider $x_n := n + 1/n$ and $y_n := n + 1/(n+1)$ for $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ but $f(x_n) - f(y_n) = (x_n - y_n)^3 + 3x_n y_n (x_n - y_n) = 1/n^2(n+1)^2 + 3(n^2+1)(n^2+n+1)/n^2(n+1)^2 \rightarrow 3$ as $n \rightarrow \infty$. ■

(v) f is U.C. Indeed, $|f(x) - f(y)| = \frac{|x^2 - y^2|}{(1+y^2)(1+x^2)} \leq \left(\frac{|x|}{1+x^2} + \frac{|y|}{1+y^2}\right)|x-y| \leq M|x-y|$ for some $M > 0$.

3. Suppose that f is uniformly continuous on A . If (x_n) be a Cauchy sequence in A , then show that $(f(x_n))$ is a Cauchy sequence. Show by an example that if f is merely continuous on A then $(f(x_n))$ may not be a Cauchy sequence.

Solution: Let (x_n) be a Cauchy sequence in A . Choose $\epsilon > 0$. Then there exists $\delta > 0$ such that $x, y \in A$ and $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Now there exists $p \in \mathbb{N}$ such that $|x_n - x_m| < \delta$ for all $n, m \geq p$. Consequently, $|f(x_n) - f(x_m)| < \epsilon$ for all $n, m \geq p$. Hence $(f(x_n))$ is a Cauchy sequence. ■

Example: Consider $f : (0, 2) \rightarrow \mathbb{R}$ given by $f(x) := 1/x$. Then $(1/n)$ is Cauchy but $(f(1/n))$ is not. ■

4. If f is uniformly continuous on A and A is bounded then show that f is bounded. Show that a continuous function on a bounded set may not be bounded.

Solution: If possible, suppose that f is unbounded. Then there exists a sequence (x_n) in A such that $|f(x_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Since (x_n) is bounded, without loss of generality (Bolzano-Weierstrass), we assume that x_n is a convergent sequence. Then $(f(x_n))$ is a convergent sequence and therefore is bounded - which is a contradiction. Hence f is bounded. ■

Example: Consider $f : (0, 1) \rightarrow \mathbb{R}$ given by $f(x) := 1/x$. Then f is continuous on $(0, 1)$ and is unbounded. ■

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) := x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) := 0$. Show that f is differentiable on \mathbb{R} . Is f' a continuous function?

Solution: That f is differentiable on \mathbb{R} is easy to show. We have $f'(x) = 2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$ and $f'(0) = 0$. This shows that f' is not continuous at 0. ■

6. Let $f : (a, b) \rightarrow \mathbb{R}$ be such that $|f(x+h) - f(x)| \leq C|h|^m$ for all $x, x+h \in (a, b)$, where C is a constant and $m > 1$ is a rational number. Show that f is differentiable on (a, b) and compute $f'(x)$ for $x \in (a, b)$.

Solution: By sandwich theorem, $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ exists and is equal to 0. Hence $f'(x) = 0$ for all $x \in (a, b)$. ■

7. If $f : (a, b) \rightarrow \mathbb{R}$ is differentiable at $c \in (a, b)$ then show that $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c-h)}{2h}$ exists and is equal to $f'(c)$. Is the converse true?

Solution: If f is differentiable at c then $f(c+h) - f(c-h) = f(c+h) - f(c) + f(c) - f(c-h)$ gives the desired result. For the converse, consider $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = |x|$. Then at $c = 0$ the given condition holds but f is not differentiable at 0. ■

8. Let $f : (a, b) \rightarrow \mathbb{R}$ and $c \in (a, b)$. Show that the following are equivalent.

- (i) f is differentiable at c .
- (ii) There exist $L \in \mathbb{R}, \delta > 0$ and a function $g : (-\delta, \delta) \rightarrow \mathbb{R}$ such that $f(c+h) = f(c) + Lh + hg(h)$ for all $h \in (-\delta, \delta)$ and $\lim_{h \rightarrow 0} g(h) = 0$.
- (iii) There exists $L \in \mathbb{R}$ such that $\lim_{h \rightarrow 0} \frac{|f(c+h) - f(c) - Lh|}{|h|} = 0$.

Solution: Trivial. For example, consider the implication (i) \Rightarrow (ii). Define $g(h) = \frac{f(x+h)-f(c)}{h} - f'(c)$ if $h \neq 0$ and $g(0) = 0$. Then we have (ii). The implication (ii) \Rightarrow (iii) \Rightarrow (i) is easy. ■

9. Given $g(x) := f(\frac{2x-1}{x+1})$ and $f'(x) = \sin(x^2)$, compute $g'(x)$.

Solution: Apply chain rule. $g'(x) = 3 \sin((\frac{2x-1}{x+1})^2)/(x+1)^2$. ■

10. Use the theorem on the derivative of inverse function to find the derivative of $g(x) := (f(x))^r$, where $r > 0$ is rational, f is differentiable and $f(x) \geq 0$.

Solution: Let $r = m/n$ with $\gcd(m, n) = 1$. Then $g(x) = (h \circ k \circ f)(x)$, where $h, k : [0, \infty) \rightarrow \mathbb{R}$ are given by $h(x) = x^m$ and $k(x) = x^{1/n}$. Apply composition rule and evaluate k' from the derivative of $k^{-1}(y) = y^n$. This gives $g'(x) = r(f(x))^{r-1} f'(x)$. ■

11. Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable. Show that f' has the Intermediate Value Property on $[a, b]$, that is, f' assumes all values between $f'(a)$ and $f'(b)$.

Solution: Suppose that $f'(a) < \lambda < f'(b)$. Define $g(x) = f(x) - \lambda x$. Note that g is continuous on $[a, b]$ and so attains its maximum at $c \in [a, b]$. Since $g'(a) < 0$ and $g'(b) > 0$, we see that $c \in (a, b)$. Consequently, $g'(c) = 0$ gives $f'(c) = \lambda$. ■

12. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that $f'(c) = \lim_{n \rightarrow \infty} [n(f(c + 1/n) - f(c))]$. Is the converse true?

Solution: Since $f'(c)$ exists, $\lim_{n \rightarrow \infty} [n(f(c + 1/n) - f(c))] = \lim_{n \rightarrow \infty} \frac{f(c + 1/n) - f(c)}{1/n} = f'(c)$.

Converse is false. Consider $f(x) = |x|$. Then for $c = 0$, $\lim_{n \rightarrow \infty} n[f(1/n) - f(0)] = 1$ but $f'(0)$ does not exist. ■

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