QUESTIONS:

1. (a) Let $a_1 = 2.5$. For each natural n, define $a_{n+1} = 0.9a_n + 2022$. Show that the sequence (a_n) converges and find the limit. [2]

Soln.: Method-1

A few calculations:

n	a_n
1	2.5
2	2024.25
3	3843.825
4	5481.4425
5	6955.2982
6	8281.7683
7	9475.5914
8	10550.0322
9	11517.0289
10	12387.3260

makes one suspect that a_n is monotonically increasing. A "high-school" rush of taking $\ell = \lim a_n$ on $a_{n+1} = 0.9a_n + 2022$ yields: $\ell = 0.9\ell + 2022$, i.e. $\ell = 20,220$. This doesn't prove the answer is 20,220. Do you realize why?

Check $a_2 = 2024.25 > 2.5 = a_1$. Assume $a_{k+1} > a_k$. Now, $a_{k+2} = 0.9a_{k+1} + 2022 > 0.9a_k + 2022 = a_{k+1}$. By Principle of Mathematical Induction, (a_n) is a monotonically increasing sequence. Check $a_1 < 2022$. Assume $a_k < 20,220$. Now, $a_{k+1} = 0.9a_k + 2022 < 0.9(20,220) + 2022 = 20,220$. By Induction, (a_n) is bounded above by 20,220. (a_n) being an increasing and bounded-above sequence converges. High-School Rush: $\lim a_{n+1} = 0.9 \lim a_n + 2022$ implies $\ell = 20,220$.

Method-2

By iterating, check and guess that

$$\begin{aligned}
a_1 &= 2.5,\\
a_2 &= (0.9)(2.5) + 2022,\\
a_3 &= (0.9)^2(2.5) + (1 + 0.9)2022,\\
a_4 &= (0.9)^3(2.5) + (1 + 0.9 + 0.9^2)2022,\\
\ldots &= \ldots\\
a_n &= (0.9)^{n-1}(2.5) + (1 + 0.9 + \cdots + 0.9^{n-2})2022 \text{ for } n \geq 3.
\end{aligned}$$

Applying induction, the given sequence for all $n \geq 3$ satisfies

$$\begin{aligned}
a_n &= 2.5b_n + 2022c_n\\
\text{where } b_n &= (0.9)^{n-1}\\
\text{and } c_n &= 1 + 0.9 + \cdots + 0.9^{n-2}
\end{aligned}$$

Both $\lim b_n$ & $\lim c_n$ exist and $\lim b_n = 0$ and $\lim c_n = \frac{1}{1-0.9} = 10$.

By elementary rules of limits, $\lim a_n = 2.5(0) + 2022(10) = 20,220$.

(b) Using axioms for $\mathbb{R}$, prove that the function $\lambda : \mathbb{R} \to \mathbb{R}$ given by $\lambda(x) = x^3$ is injective.[3]

Soln.: Suppose λ is NOT injective, i.e., there exist $x, y \in \mathbb{R}$, $x \neq y$ with $\lambda(x) = \lambda(y)$, i.e.,

$x^3 = y^3$.

$$\begin{array}{rcl} & x^3 = y^3 & \\ \Rightarrow & x^3 - y^3 = 0 & \text{Add } (-y^3) \text{ to both sides} \\ \Rightarrow & (x-y)(x^2+xy+y^2) = 0 & \text{Use distrib. of multip. over add.)} \\ \Rightarrow & x - y = 0\text{–OR–}(x^2+xy+y^2) = 0 & \text{Recall: } ab = 0 \Rightarrow a = 0 \text{ OR } b = 0 \end{array}$$

Case (I) $x - y = 0$. Add y to both sides and using associativity and the fact that $-y$ is the additive inverse of y, we get $(x-y)+y = 0+y \Rightarrow (x+(-y))+y = y \Rightarrow x+((-y)+y) = y \Rightarrow x+0 = y \Rightarrow x = y$, a contradiction to our assumption that $x \neq y$. Case (II) $x^2+xy+y^2 = 0$ Method-1:

Next, recall that we have proved in class/tutorial that for a real number $a \neq 0$, $a^2 > 0$, and also, $a^2 = 0 \Leftrightarrow a = 0$. Using the latter, we have that for two real numbers a, b, $a^2 + b^2 \geq 0$ and $a^2 + b^2 = 0$ implies $a = b = 0$.

Completing the square in $x^2+xy+y^2 = (x+\frac{1}{2}y)^2 + \frac{3}{4}y^2$. Take $a = (x+\frac{1}{2}y)$ and $b = \sqrt{\frac{3}{4}}y$ and from the above work, we have $a^2+b^2 = x^2+xy+y^2 = 0$. So we get $a = b = 0$, i.e. $x + \frac{1}{2}y = \sqrt{\frac{3}{4}}y = 0$. From this conclude $y = 0$ and then $x = 0$.

Method-2:

Case (i) $0 \leq x < y$: Then $0 \leq x^2 + xy$ and $0 < y$ implies $0 < y^2$. So that $0 < x^2+xy+y^2$, i.e., a contradiction to first-step.

Case (ii) $x < 0 \leq y$:

Case(.) $y + x > 0$: $x^2+xy+y^2 = x^2+(x+y)y > 0$, a contradiction to first-step,

Case(..) $y + x = 0$: $x^2+xy+y^2 = x^2+(x+y)y > 0$, a contradiction to first-step.

Case(...) $y + x < 0$: : $x^2+xy+y^2 = (x+y)x+y^2 > 0$, a contradiction to first-step.

Case (iii) $x < y \leq 0$: Similar to Case (i).

Method-3:

Case (i) $y = 0$: Then $x^2 = 0$ implies $x = 0$. A contradiction to $x \neq y$.

Case (ii) $y \neq 0$: Then by dividing $x^2+xy+y^2 = 0$ by $y^2 \neq 0$, get $\lambda^2+\lambda+1 = 0$ where $\lambda = \frac{x}{y}$.

If you follow completing the square, as in Method-1 and prove that there are no solutions $x^2+xy+y^2 = 0$, then you get full marks, i.e. Note: If you use discriminant of a quadratic (you are not using axioms and immediately derived properties), you lose 1/2 mark, i.e. you get only 0.5 marks, instead of 1.

2. (a) Show that the series $\sum \frac{(n!)^2}{(2n+2)!}$ converges. [2]

Soln.: Write $a_n = \frac{(n!)^2}{(2n+2)!}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{((n+1)!)^2}{(2(n+1)+2)!}}{\frac{(n!)^2}{(2n+2)!}} = \frac{(n+1)^2}{(2n+3)(2n+4)}$$

Now taking limit as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{n\left(2+\frac{3}{n}\right) n\left(2+\frac{4}{n}\right)} \right) \\
&= \lim_{n\to\infty} \left(\frac{1+\frac{2}{n}+\frac{1}{n^2}}{\left(\frac{3}{n}+2\right)\left(\frac{4}{n}+2\right)} \right) \\
&= \lim_{n\to\infty} \left(\frac{1+\frac{2}{n}+\frac{1}{n^2}}{\left(\frac{3}{n}+2\right)\left(\frac{4}{n}+2\right)} \right) \\
&= \frac{\lim_{n\to\infty} 1+\frac{2}{n}+\frac{1}{n^2}}{\lim_{n\to\infty}\left(\frac{3}{n}+2\right)\left(\frac{4}{n}+2\right)} \\
&= \frac{1}{4} < 1.
\end{aligned}$$

Therefore, by **Ratio Test**, the series converges absolutely. As $a_n > 0$ for all n, the series converges.

Soln.: Using Comparison test:

Note that for each $n \in \mathbb{N}$,

$$0 < \frac{(n!)^2}{(2n+2)!} = \frac{(n!)^2}{(2n+2)(2n+1)(2n)!} < \frac{(n!)^2}{(2n)!}.$$

Consider $b_n = \frac{(n!)^2}{(2n)!}$.

Then

$$\begin{aligned}
\frac{b_{n+1}}{b_n} &= \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} \\
&= \frac{\frac{(n+1)^2(n!)^2}{(2n+2)(2n+1)(2n)!}}{\frac{(n!)^2}{(2n)!}} \\
&= \frac{(n+1)^2(n!)^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(n!)^2} \\
&= \frac{(n+1)^2}{2(n+1)(2n+1)}
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2(n+1)(2n+1)} = \frac{1}{4} < 1.$$

Therefore, by Ratio Test, the series $\sum b_n$ converges, and hence by **Comparison test**, the series $\sum a_n$ converges.

(b) Two sequences of real numbers (a_n) and (b_n) satisfy $a_n = b_n$ for all $n \geq k$, for some natural number k. Show that $\sum a_n$ converges if and only if $\sum b_n$ converges. [3]

Soln.: Let s_n denote the sequence of partial sums of the series $\sum a_n$ and let t_n denote the sequence of partial sums of the series $\sum b_n$. Further let $s_{k-1} = \alpha$ and $t_{k-1} = \beta$. Observe that for all $n \geq k$, $s_n - \alpha = t_n - \beta$. (Since $a_n = b_n$, for all $n \geq k$.)

Assume $\sum b_n$ converges. By definition, $\lim t_n$ exists. Using this above, $\lim s_n = \alpha - \beta + \lim t_n$ also exists. Hence $\sum a_n$ converges. Similarly, we can say that $\sum b_n$ converges if $\sum a_n$ converges.

Soln.:

We have $a_n = b_n$ for all $n \geq k$, for some natural number k. Let $S_n = \sum_{m=1}^{n} a_m$ and $T_n = \sum_{m=1}^{n} b_m$. Then, for $n > m \geq k$, we have

$$\begin{aligned} |S_n - S_m| &= |a_{m+1} + a_{m+2} + ... + a_n| \\ &= |b_{m+1} + b_{m+2} + ... + b_n| \\ &= |T_n - T_m| \end{aligned}$$

Now, suppose $\sum a_n$ converges. Then by the Cauchy Criterion, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|S_n - S_m| < \epsilon \quad \text{for all} \quad n > m \geq N.$$

Consider $M = \max(N, k)$. Then for every $\epsilon > 0$, we have

$$|T_n - T_m| = |S_n - S_m| < \epsilon \quad \text{for all} \quad n > m \geq M.$$

Therefore, by Cauchy criterion, the sequence of partial sums $\sum b_n$ is convergent.
Similarly, we can say that $\sum a_n$ converges if $\sum b_n$ converges.

3. (a) Let $h : \mathbb{R} \to \mathbb{R}$ be a function. For each of the statements given below, prove it if it is true. If the statement is false, give a counter–example.

   i. If h is continuous, for every unbounded sequence of real numbers (x_n) the sequence $(h(x_n))$ is unbounded. [1]
   Soln.: Counter Example-1: The constant function $h(x) \equiv 2.5$ is continuous, the sequence $x_n = n$ is unbounded but the sequence $h(x_n) = 2.5$ is bounded.
   Counter Example-2: The sine function $h(x) = \sin(x)$ is continuous, the sequence $x_n = 2n \cdot \pi$ is unbounded, but the sequence $h(x_n) = 0$ is bounded.
   Any valid counter example where h is 'known' to be continuous from class, where unboundedness of x_n is known from class and boundedness of $h(x_n)$ is evident will get a mark.

   ii. If h is continuous, for every bounded sequence of real numbers (y_n) the sequence $(h(y_n))$ is bounded. [2]
   Soln.: The statement is true. If (y_n) is a bounded sequence, there exists a real number $B > 0$ such that $|y_n| \leq B$. Consider the restriction of h to the interval $[-B, B]$, $h_B : [-B, B] \to \mathbb{R}$ (defined via the usual $h_B(x) = h(x)$). h_B is continuous as it is the restriction of a continuous function. Range(h_B) is bounded as h_B is continuous and its domain of definition is a compact interval. Therefore, $(h(y_n)) \subset$ Range(h_B) is bounded.

   (b) Let $h : [0, 1] \to \mathbb{R}$ be a continuous function. If (z_n) is a cauchy sequence in $[0, 1]$, show that the sequence $(h(z_n))$ converges. [2]

   Soln.: The statement is true. From the proposition that every cauchy sequence converges, the given sequence z_n being cauchy converges to a real number z, say. From the inequalities $0 \leq z_n \leq 1$, and $z_n \to z$, we get $0 \leq z \leq 1$. Conclude that $z \in [0, 1]$. By sequential criterion for continuity $h(z_n) \to h(z)$.

4. (a) Let $p : \mathbb{R} \to \mathbb{R}$ be the polynomial function $p(x) = \frac{1}{2}x^5 - 400x^4 + 30x^3 - 2000x^2 + x - 100$. Show that there exists a $c \in \mathbb{R}$ such that $p(c) = 0$. [2]

Soln.: Note the following inequalities:

$$
\begin{aligned}
&\frac{1}{5}\cdot\frac{1}{2}x^5 \\
&\quad = \frac{x}{10}x^4 > 400x^4 \text{ for every } x > 4,000. \\
&\quad = \frac{x^2}{10}x^3 > -30x^3 \text{ for every } x > 0. \\
&\quad = \frac{x^3}{10}x^2 > 2000x^2 \text{ for every } x > \sqrt[3]{20,000}. \\
&\quad = \frac{x^4}{10}x^1 > -x \text{ for every } x > 0. \\
&\quad = \frac{x^5}{10} > 100 \text{ for every } x > \sqrt[5]{1,000}.
\end{aligned}
$$

Adding these 5 inequalities, $5\cdot\frac{1}{5}\cdot\frac{1}{2}x^5 = \frac{1}{2}x^5 > 400x^4 - 30x^3 + 2000x^2 - x + 100$ for every $x > \max(4000, 0, \sqrt[3]{20,000}, 0, \sqrt[5]{1,000}) = 4000$. Thus we have proved, for instance, that $p(4001) > 0$. Or using any method, find a real number β and verify by calculation that $p(\beta) > 0$.

By substitution $-100 = p(0) < 0$ or any other valid α such that $p(\alpha) < 0$. Mention p is continuous and apply Intermediate value theorem to get $p(c) = 0$ for some $c \in (0, 4001)$, in our calculation or $c \in (\alpha, \beta)$, assuming your $\alpha < \beta$.

Some calculations for your reference are listed below. These are not expected on your test booklet. But they are given to give an idea of where p is positive and where it is negative. This table is also useful for graders as a quick reference to check some of the calculations of the students.

x	p(x)
-10000	-54000030200000010100.0
-2000	-22400248000002100.0
-1000	-900032000001100.0
-500	-40629250000600.0
-100	-45050000200.0
-50	-2665000150.0
-10	-4280110.0
-1	-2531.5
0	-100.0
0.5	-620.73438
1	-2468.5
10	-4120090.0
20	-62960080.0
50	-2345000050.0
100	-34990000000.0
200	-479839999900.0
300	-2024369999800.0
500	-9371749999600.0
700	-11995689999400.0
799.93	-254790015.60049
799.93124	-946729.14402
799.93125	1100406.92678
799.94	1792422937.23531
800	14080000700.0
1000	100028000000900.0
2000	9600232000001900.0
4001	4101397710979715 31.5

Method-2 Every odd degree polynomial, whose co-efficients are real numbers, has a real root. This was proved as a tutorial problem. Hence p has a real root.

(b) Let $q(x) = \sqrt{2022} + (4x-1)(3x-1)(2x-1)(x^2-1)(x+2)(x+3)(x+4)$. Show that there exist at least 7 real numbers α_i such that $q'(\alpha_i) = 0$ for $i \in \{1, 2, \ldots, 7\}$. [3]

Soln.: Method-1:

Define $\{c_i\}_1^8$ as follows:

$$c_1 = -4 < c_2 = -3 < c_3 = -2 < c_4 = -1 < c_5 = \frac{1}{4} < c_6 = \frac{1}{3} < c_7 = \frac{1}{2} < c_8 = 1.$$

Note that $q(c_i) = \sqrt{2022}$ for every $1 \le i \le 8$. If the student finds only a couple, or three or four etc. of c_i, then stepmarks may be given proportionately.

We can apply Rolle's Theorem as q is continuous and differentiable. In particular, applying Rolle's Theorem to the 7 intervals $[c_i, c_{i+1}]$ for $i \in \{1, 2, \ldots, 7\}$, we get the required 7 real numbers α_i such that $q'(\alpha_i) = 0$. Note that while applying Rolle's Theorem, we have used the observation that restrictions of a continuous and differentiable function are continuous and differentiable. Method-2:

Define $\{c_i\}_1^8$ as follows:

$$c_1 = -4 < c_2 = -3 < c_3 = -2 < c_4 = -1 < c_5 = \frac{1}{4} < c_6 = \frac{1}{3} < c_7 = \frac{1}{2} < c_8 = 1.$$

Need to show, by calculations that $q'(c_1), q'(c_2), q'(c_3), q'(c_4), \ldots q'(c_8)$ are alternately negative, positive, negative, positive, ... If the student finds only a couple, or three or four etc. of c_i, then stepmarks may be given proportionately.

We can apply Intermediate Value Theorem to q' as q' is continuous. Now, applying Intermediate Value Theorem to the 7 intervals $[c_i, c_{i+1}]$ for $i \in \{1, 2, \ldots, 7\}$, we get the required 7 real numbers α_i such that $q'(\alpha_i) = 0$. Note that while applying Intermediate Value Theorem, we have used the observation that restrictions of a continuous function are continuous.

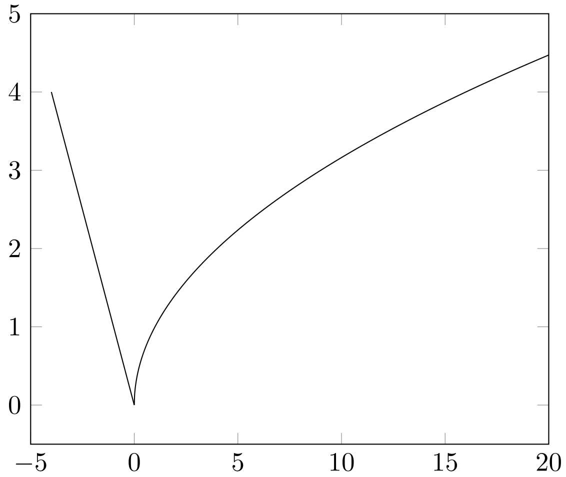
5. Define $g : \mathbb{R} \to \mathbb{R}$ as:

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } x \geq 0. \end{cases}$$

(a) Using Weierstrass' Criterion, show that g is continuous at $x = 0$. [3]

Soln.:

Graph of g:



Preliminary Observations: Note that $g(0) = 0$. We give examples of different values of ϵ and explore to propose suitable values of δ such that if $|x - 0| < \delta$, then $|g(x) - g(0)| = |g(x)| < \epsilon$.

i. Suppose $\epsilon = 4$ is given. To make $|g(x)| < 4$, guess from the graph the two trivia: If x is negative, then $x > -4$ ensures $g(x) = -x < 4$. If x is positive, then $x < 16$ ensures $g(x) = \sqrt{x} < 4$. Thus $\delta = 4 = \min(|-4|, 16)$ would "work" for $\epsilon = 4$. Of course, any number smaller than 4 would "work" as delta.

ii. Suppose $\epsilon = 2$ is given. To make $|g(x)| < 2$, guess from the graph the two trivia: If x is negative, then $x > -2$ ensures $g(x) = -x < 2$. If x is positive, then $x < 4$ ensures $g(x) = \sqrt{x} < 2$. Thus $\delta = 2 = \min(|-2|, 4)$ would "work" for $\epsilon = 2$. Of course, any number smaller than 2 would "work" as delta.

iii. Suppose $\epsilon = 0.5$ is given. To make $|g(x)| < 0.5$, guess from the graph the two trivia: If x is negative, then $x > -0.5$ ensures $g(x) = -x < 0.5$. If x is positive, then $x < 0.25 = 0.5^2$ ensures $g(x) = \sqrt{x} < 0.5$. Thus $\delta = .25 = \min(|-0.5|, 0.25)$ would "work" for $\epsilon = 2$. Of course, any number smaller than .25 would "work" as delta.

iv. Suppose $\epsilon = 0.25$ is given. To make $|g(x)| < 0.25$, guess from the graph the two trivia: If x is negative, then $x > -0.25$ ensures $g(x) = -x < 0.25$. If x is positive, then $x < 0.0625 = 0.25^2$ ensures $g(x) = \sqrt{x} < 0.25$. Thus $\delta = .0625 = \min(|-0.25|, 0.0625)$ would "work" for $\epsilon = .25$. Of course, any number smaller than .0625 would "work" as delta.

What did you learn from the above observations? From the list of observations above, one can guess that $\delta_1 = \epsilon$ "works" for negative values of x and $\delta_2 = \epsilon^2$ "works" for non–negative values of x. Therefore take $\delta = \min(\delta_1, \delta_2) > 0$. Next assume x is any real number satisfying $|x| < \delta$. Then: $|x| < \delta_1$ and $|x| < \delta_2$. The following two conclusions are true:

If x is negative, $|g(x)| = |-x| < \delta \leq \delta_1 = \epsilon$. If x is non–negative, $|g(x)| = \sqrt{x} < \epsilon$. Note that in writing $\sqrt{x} < \epsilon$ from $|x| < \delta \leq \delta_2 = \epsilon^2$ we are using that the square-root function is monotonically increasing on non–negative real numbers.

Thus we have proved continuity, using Weiserstrass criterion.

Method-2

Given a real $\epsilon > 0$, define $\epsilon_0 = \min(\epsilon, 1)$. Take $\delta = \epsilon_0^2$. Next assume x is any real number satisfying $|x| < \delta$. Then: $|x| < \epsilon_0^2$. The following two conclusions are true:

If x is negative, $|g(x)| = |-x| < \epsilon_0^2 \leq \epsilon_0 \leq \epsilon$. Note that in writing $\epsilon_0^2 \leq \epsilon_0$, we have used $\epsilon_0 \leq 1$ If x is non–negative, $|g(x)| = \sqrt{x} < \epsilon_0 < \epsilon$. Note that in writing $\sqrt{x} < \epsilon_0$ from $x^2 < \epsilon_0^2$ we are using that the square-root function is monotonically increasing on non–negative real numbers.

(b) Using the definition, show that g is not differentiable at 0.

{Warning: Do NOT use left/right–hand limits.} [2]

Soln.:

Let A be a domain in $\mathbb{R}$ and $c \in A$. Consider a function $f :\to \mathbb{R}$.

WC-lim: We say Weierstrass Criterion holds for a real number L if:
For every real $\epsilon > 0$, there exists a real $\delta > 0$ such that for all $0 < |x - c| < \delta$ it should be true that $|f(x) - f(c)| < \epsilon$.

SC-lim: We say Sequential Criterion holds for a real number L if:
For every sequence of real numbers $x_n \to c$ (with $x_n \neq c$, for every n) we have $\lim f(x_n) \to L$.

Recall that we say $\lim_{x\to c} f(x) = L$ for a real number L if WC-lim holds. You could try to prove that WC-lim and SC-lim are equivalent. Here, for the purposes of this question, can you prove: WC-lim implies SC-lim?

{Hint: Recall the proof of Weierstrass' Criterion implies Sequential Criterion (for continuity at a point) discussed in detail in class and available on Lecture Notes. A verbatim copy of that proof proves the required proposition.}

Assume g is differentiable at 0. Then $\lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = L$, exists for some real L. We apply SC-lim: Take the sequence $h_n = \frac{1}{n^2}$, note that $h_n \to 0$, that none of the $h_n = 0$. However, $\lim \frac{g(h_n)-g(0)}{h_n} = \frac{\sqrt{1/n^2}}{1/n^2} = \lim n$ does not exist as the sequence is unbounded.

Method-2

Assume g is differentiable at 0. Then $\lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = L$, exists for some real L.

Case (i) $L = -1$. Take $\epsilon = 1$. Then by our assumption of differentiability, there exists a real $\delta > 0$ such that for all $0 < |h| < \delta$, we are assured of having $|\frac{g(0+h)-g(0)}{h} - (-1)| < \epsilon = 1$. Take any $h \in (0, \delta)$ and the term $|\frac{g(0+h)-g(0)}{h} - (-1)| = \frac{1}{\sqrt{h}} + 1 > 1$, a contradiction to the assurance. Case (ii) $L \neq -1$. Take $\epsilon = \frac{|L+1|}{2} > 0$. Then by our assumption of differentiability, there exists a real $\delta > 0$ such that for all $0 < |h| < \delta$, we are assured of having $|\frac{g(0+h)-g(0)}{h} - L| < \epsilon = \frac{|L+1|}{2}$. Take any $h \in (-\delta, 0)$ and the term $|\frac{g(0+h)-g(0)}{h} - L| = |\frac{-h}{h} - L| = |L+1| > \epsilon = \frac{|L+1|}{2}$, a contradiction to the assurance.

6. Let $f : [0, 2] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} -1, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } 1 < x \leq 2. \end{cases}$$

(a) Use the definition of Riemann integrability to show that f is integrable on $[0, 2]$. [3]

Soln.: Consider a partition $\mathcal{P}$ of $[0, 2]$

$$0 = x_0 < x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots < x_n = 2$$

Define I to be the unique index in the range $\{0, 1, 2, \ldots, n-1\}$ such that $x_I \leq 1 < x_{I+1}$. Note that such I is well–defined and unique.

Let $\{\xi_i\}_{i=1}^{n}$ be any collection of tags for $\mathcal{P}$. Note that $f(\xi_i) = -1$ for all $1 \leq i \leq I$, that $f(\xi_i) = 1$ for $I + 2 \leq i \leq n$.

Therefore, the Riemann sum

$$\begin{aligned} S(f, \mathcal{P}) &= \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{I} f(\xi_i)(x_i - x_{i-1}) + f(\xi_{I+1})(x_{I+1} - x_I) + \sum_{i=I+2}^{n} f(\xi_i)(x_i - x_{i-1}) \\ &= (-1)(x_I - x_0) + f(\xi_{I+1})(x_{I+1} - x_I) + (1)(x_n - x_{I+1}) \\ &= (-1)(x_I - 0) + f(\xi_{I+1})(x_{I+1} - x_I) + (1)(2 - x_{I+1}). \end{aligned}$$

Rewriting $(-1)x_I$ as $(-1) + (1 - x_I)$ and $2 - x_{I+1}$ as $1 - (x_{I+1} - 1)$.

$$\begin{aligned} |S(f, \mathcal{P}) - 0| &= |(-1) \cdot x_I + f(\xi_{I+1})(x_{I+1} - x_I) + (1) \cdot (2 - x_{I+1})| \\ &= |(-1) + (1 - x_I) + f(\xi_{I+1})(x_{I+1} - x_I) + (1) \cdot (1 - (x_{I+1} - 1))| \\ &= |(1 - x_I) + f(\xi_{I+1})(x_{I+1} - x_I) + (1 - x_{I+1})| \\ &\leq |(1 - x_I)| + |f(\xi_{I+1})(x_{I+1} - x_I)| + |(1 - x_{I+1})| \\ &\leq |(x_{I+1} - x_I)| + |f(\xi_{I+1})(x_{I+1} - x_I)| + |(x_{I+1} - x_I)| \end{aligned}$$

Since $1 < \xi_{I+1}$, we have $f(\xi_{I+1}) = 1$. Therefore,

$$|S(f, \mathcal{P})| = |(x_{I+1} - x_I)| + |(x_{I+1} - x_I)| + |(x_{I+1} - x_I)|.$$

Now, given any real $\epsilon > 0$, choose $\delta = \frac{\epsilon}{3}$. Let $\mathcal{P} = \{[x_{i-1}, x_i] \mid 0 \leq i \leq n\}$ be a partition of $[0, 2]$ such that $\|\mathcal{P}\| < \delta$. Therefore $|x_i - x_{i-1}| < \delta$, for all i.

Thus,

$$|S(f, \mathcal{P}) - 0| = |S(f, \mathcal{P})| \leq \delta + \delta + \delta < 3\delta = \epsilon.$$

Hence, f is Riemann integrable on $[0, 2]$.

(b) Explain why f is Riemann integrable on $[0, x]$, for every $x \in [0, 2]$. [0.5]

Soln.: **Recall:** If f is integrable on an interval S, then it is integrable on every subinterval $I \subset S$.

(c) Define $F : [0, 2] \to \mathbb{R}$ via $F(x) = \int_0^x f$. Show that F is continuous on $[0, 2]$. [1.5]

Soln.: Let $y, z \in [0, 2]$ with $y < z$. Then,

$$F(z) = \int_0^z f = \int_0^y f + \int_y^z f = F(y) + \int_y^z f.$$

Therefore, $F(z) - F(y) = \int_y^z f$. We know that $-1 \leq f(x) \leq 1$ for all $x \in [0, 2]$. Therefore,

$$(-1)(z - y) \leq \int_y^z f \leq (1)(z - y),$$

and hence $|F(z) - F(y)| = |\int_y^z f| \leq |z - y|$.

To check continuity of F at $x = a \in [0, 2]$: Let $\epsilon > 0$. Choose $\delta = \epsilon$. Then for $x \in (a - \delta, a + \delta) \cap [0, 2]$,

$$|F(x) - F(a)| \leq |x - a| < \delta = \epsilon.$$

Therefore, F is continuous at a.

For better understanding of F, show that $F(x) = -x$ for $x \in [0, 1]$ and $F(x) = x - 2$ for $x \in [1, 2]$.

7. (a) Write down the radii of convergence of each of the power series given below. You do not have to justify your answer for this part.

i. $\sum_1^\infty (9n^2)^n x^n$
ii. $\sum_1^\infty (8n)^n x^n$
iii. $\sum_1^\infty 7^n x^n$
iv. $\sum_1^\infty \frac{x^n}{(6n)!}$ [2]

Soln.:

i. $\sum_1^\infty (9n^2)^n x^n, \quad r = 0$
ii. $\sum_1^\infty (8n)^n x^n, \quad r = 0$
iii. $\sum_1^\infty 7^n x^n, \quad r = \frac{1}{7}$
iv. $\sum_1^\infty \frac{x^n}{(6n)!}, \quad r = \infty$.

(b) Consider $\theta : (0, \infty) \to (0, \infty)$ given by $\theta(x) = \frac{1}{x^2} + \frac{1}{x}$. Show that θ is bijective and that θ^{-1} is continuous. [3]

Soln.: To show that θ is one-to-one, suppose that $\theta(x_1) = \theta(x_2)$ for some $x_1, x_2 \in (0, \infty)$. Then

$$\frac{1}{x_1^2} + \frac{1}{x_1} = \frac{1}{x_2^2} + \frac{1}{x_2}$$
$$\implies \left(\frac{1}{x_1} + \frac{1}{2}\right)^2 - \frac{1}{4} = \left(\frac{1}{x_2} + \frac{1}{2}\right)^2 - \frac{1}{4}$$
$$\implies \left(\frac{1}{x_1} + \frac{1}{2}\right) = \begin{cases} \left(\frac{1}{x_2} + \frac{1}{2}\right) & \text{OR} \\ -\left(\frac{1}{x_2} + \frac{1}{2}\right) \end{cases}$$

As $x_2 > 0$, $-\left(\frac{1}{x_2} + \frac{1}{2}\right) < 0$. Therefore,

$$\left(\frac{1}{x_1} + \frac{1}{2}\right) = \left(\frac{1}{x_2} + \frac{1}{2}\right) \implies x_1 = x_2.$$

Therefore θ is injective.

To show that θ is surjective, we need to show that for any $y \in (0, \infty)$, there exists an $x \in (0, \infty)$ such that $\theta(x) = y$. Let $y \in (0, \infty)$. Then we can solve the equation $\theta(x) = y$ for x to get

$$x = \frac{1 \pm \sqrt{1 + 4y}}{2y}$$

Since $y > 0$, $\frac{1 + \sqrt{1+4y}}{2y} > 0$, so θ is surjective.

Therefore, θ is bijective. And the inverse function is given by

$$\theta^{-1}(y) = \frac{1 + \sqrt{1 + 4y}}{2y}.$$

We will show that θ^{-1} is continuous using Sequential Criterion:

Let (y_n) be a convergent sequence in $(0,\infty)$, such that $y_n \to y$ and $y \in (0,\infty)$. Consider the sequence $x_n = \frac{1+\sqrt{1+4y_n}}{2y_n}$. Since $y_n \neq 0$ and $y \neq 0$, by applying ratio of limits, we get

$$\begin{aligned}\lim x_n &= \lim \frac{1+\sqrt{1+4y_n}}{2y_n} \\ &= \frac{\lim(1+\sqrt{1+4y_n})}{\lim(2y_n)} \\ &= \frac{1+\sqrt{1+4y}}{2y}\end{aligned}$$

which lies in $(0,\infty)$. Hence the sequence $(x_n) = (\theta^{-1}(y_n))$ is convergent. As (y_n) is arbitrary, by sequential criterion, θ^{-1} is continuous.

Soln.: (Alternate-1: Continuity Using Inverse function theorem):

Let $0 < x < y$ be real numbers. Then

$$\begin{aligned}x < y \implies & \quad \frac{1}{y} < \frac{1}{x} \\ \implies & \quad \frac{1}{y^2} + \frac{1}{y} < \frac{1}{x^2} + \frac{1}{x} \\ \implies & \quad \theta(y) < \theta(x).\end{aligned}$$

Thus, θ is a strictly decreasing function. Clearly, θ is continuous. Therefore, by **Continuous Inverse Function Theorem**, θ^{-1} is monotonic and continuous.

Note: You still need to prove bijectivity, as shown earlier.

Soln.: (Injectivity using derivatives):

It is enough to show that θ is strictly monotonic. To show this, consider $\theta'(x) = -\frac{2}{x^3} - \frac{1}{x^2}$. Note that $\theta'(x) < 0$, for all $x \in (0,\infty)$. This proves θ is injective.

Note: You still need to prove surjectivity, continuity etc.

—–Paper Ends—–