INSTRUCTIONS: This is a 90 minutes' test. Show all steps. There are 3 questions each worth 5 marks and one bonus question. Answer all questions in designated space or over the extra pages towards the end of the booklet. Use supplementary sheets ONLY for rough work. DO NOT TIE supplementary sheets to this answer booklet.

1. (a) State whether the following statements are true or false.

   (i) If a set $S \subset \mathbb{R}$ has an upper bound, then $S$ has infinitely many upper bounds. **Soln.:** TRUE [0.5]

   (ii) For a set $S \subset \mathbb{R}$, it is known that for every $s \in S$, there exists a $t \in S$ such that $t < s$. Then, $S$ has NO lower bound. **Soln.:** FALSE [0.5]

   (b) Prove the Generalized Archimedean Property: Let $(\delta_n)$ be a sequence of positive real numbers such that $\lim \delta_n = 0$. Show that for any given real number $r$, there exists an $N \in \mathbb{N}$ such that $\frac{1}{\delta_N} > r$. [1]

   **Soln.:**

   Case (a) If $r \leq 0$, take $N = 1$ and since by hypothesis we have $\delta_1 > 0$, $\frac{1}{\delta_1} > 0 \geq r$.

   Case (b) If $r > 0$, set $\epsilon = \frac{1}{r}$ in the definition of convergence of the sequence $(\delta_n)$. This gives a natural number $N$, such that for all $n \geq N$, we have $|\delta_n - 0| < \frac{1}{r} = \epsilon$. For $n = N$, since $\delta_N > 0$, we get $\frac{1}{\delta_N} > r$.

   (c) Let $S = \left\{-\frac{9}{m} + 1 + \frac{1}{n} \mid m, n \in \mathbb{N}\right\}$. Find $\inf S$ and $\sup S$. Justify your answer. [3]

   **Soln.:**

   Part (i) Upper bound: For any natural numbers $m, n$, it is true that $-\frac{9}{m} + 1 + \frac{1}{n} \leq 0 + 1 + 1$ as $n \geq 1$ and $m > 0$. Thus 2 is an upper bound for $S$.

   Every real number greater than or equal to 2 is an upper bound for $S$. (No real number less than 2 is an upper bound: See Part (iv) below)

   Part (ii) Lower bound: For any natural numbers $m, n$, it is true that $-9 + 1 + 0 \leq -\frac{9}{m} + 1 + \frac{1}{n}$ as $n > 0$ and $m \geq 1$. Thus $-8$ is a lower bound for $S$.

   Every real number less than or equal to $-8$ is a lower bound for $S$. (No real number greater than $-8$ is a lower bound. See Part (v) below)

   Part (iii) Clearly $S$ is nonempty. For instance $-9 + 1 + 1 = -7$ is a member of $S$. And from parts (i), (ii) and (iii) we know that $\sup S$ and $\inf S$ exist.

   Part (iv) If $\ell$ is any real number less than 2, then take $\epsilon = 2 - \ell > 0$. By Archimedean Property, there exists a natural number $M > \frac{9}{\epsilon}$. This implies $-\frac{9}{M} > -\epsilon$, i.e., $\alpha := -\frac{9}{M} + 1 + \frac{1}{1} > 2 - \epsilon = \ell$. The element $\alpha \in S$ and is greater than $\ell$, proving that $\ell$ is NOT an upper bound for $S$. This shows that 2 is the least upper bound, the $\sup S$.

   Part (v) If $g$ is any real number greater than $-8$, then take $\epsilon = g - (-8) > 0$. By Archimedean Property, there exists a natural number $N > \frac{1}{\epsilon}$. This implies $\frac{1}{n} < \epsilon$, i.e., $\beta := -\frac{9}{1} + 1 + \frac{1}{n} < -8 + \epsilon = g$. The element $\beta \in S$ and is less than $g$, proving that $g$ is NOT a lower bound for $S$. This shows that $-8$ is the greatest lower bound, the $\inf S$.

   **Alternate Solution:**

   Consider the sets $A_1 = \{\frac{-9}{m} \mid m \in \mathbb{N}\}$, $A_2 = \{1\}$, and $A_3 = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Clearly, $S = A_1 + A_2 + A_3 = \{x + y + z \mid x \in A_1, y \in A_2, z \in A_3\}$. We will find $\sup S$ and $\inf S$ by computing $\sup A_i$ and $\inf A_i$, for $i = 1, 2, 3$.

   For the set $A_2$, clearly, $\sup A_2 = 1 = \inf A_2$.

   Note that $A_1 = \{\frac{-9}{1}, \frac{-9}{2}, \frac{-9}{3}, \ldots\}$. Since $-9 \leq \frac{-9}{m}$ for all $m \in \mathbb{N}$, $-9$ is a lower bound. Now $-9 \in A_1$, there cannot be a lower bound larger than $-9$. Hence $\inf A_1 = -9$.

We claim that $\sup A_1 = 0$. Clearly, 0 is an upper bound for $A_1$. Let $\epsilon > 0$. Then by Archimedean Property, there exists $M \in \mathbb{N}$ such that $\frac{9}{\epsilon} < M$, and hence $\frac{9}{M} < \epsilon$. Therefore $-\epsilon < \frac{-9}{M} < 0$ and $\frac{-9}{M} \in A_1$. (0 is an upper bound and anything less than 0 is not an upper bound.) This proves $\sup A_1 = 0$.

Now, $A_3 = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. For each $x \in A_3$, $0 < x \leq 1$. Thus 1 is a upper bound and 0 is a lower bound for $A_3$. Now $1 \in A_3$, so anything smaller than 1 cannot be an upper bound. Therefore, $\sup A_3 = 1$. **Claim:** $\inf A_3 = 0$. Let $\epsilon > 0$. Then by Archimedean Property, there exists $N \in \mathbb{N}$ such that $\frac{1}{\epsilon} < N$, and hence $0 < \frac{1}{N} < \epsilon$. Since $\frac{1}{N} \in A_3$, we see that $\inf A_3 = 0$. (0 is a lower bound and anything greater than 0 is not a lower bound.)

**Lemma:** Let $A, B \subset \mathbb{R}$ be nonempty subsets bounded above. Define $C = \{x + y \mid x \in A, y \in B\}$. Then $C$ is nonempty subset of $\mathbb{R}$, bounded above and $\sup C = \sup A + \sup B$.

(**Note:** Proof of this lemma is not expected on the quiz, however you should try proving it now!)

Using this lemma, we see that $\sup S = \sup A_1 + \sup A_2 + \sup A_3 = 0 + 1 + 1 = 2$. Similarly, $\inf S = \inf A_1 + \inf A_2 + \inf A_3 = -9 + 0 + 1 = -8$.

2. (a) Let $(a_n)$ be the sequence given by $a_n = 1 + 2 + 3 + \cdots + n$. Does the series $\sum \frac{1}{a_n}$ converge or diverge? Justify your answer. [1]

**Soln.:** For any natural number $n$, $a_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, hence $\frac{1}{a_n} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$. Consider the partial sum

$$\begin{aligned} s_n &= \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \\ &= \left[2\left(\frac{1}{1} - \frac{1}{2}\right) + 2\left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + 2\left(\frac{1}{n} - \frac{1}{n+1}\right)\right] \\ &= 2\left(1 - \frac{1}{n+1}\right). \end{aligned}$$

We have seen in the class that $\lim \frac{1}{n} = 0$ and the sequence $\left(\frac{1}{n+1}\right)$, being a subsequence of the convergent sequence $\left(\frac{1}{n}\right)$ has the same limit. Hence, $\lim \dfrac{1}{n+1} = \lim \dfrac{1}{n} = 0$. Using the latter in the sequence of partial sums $(s_n)$, we get $\lim s_n = 2$. Therefore the series $\sum \frac{1}{a_n}$ converges and is equal to 2.

**Alternate Solution:**

For each $n \in \mathbb{N}$, we have

$$0 < \frac{1}{a_n} \leq \frac{2}{n(n+1)} \leq \frac{2}{n^2}.$$

Since $\sum \frac{2}{n^2}$ converges, by **comparison test for series**, $\sum \frac{1}{a_n}$ coverges.

**Alternate Solution:**

Let $d_n = \frac{1}{n^2}$. Then $\lim \dfrac{\frac{1}{a_n}}{d_n} = 2 \neq 0$ Alternately, $\lim \dfrac{d_n}{\frac{1}{a_n}} = \dfrac{1}{2} \neq 0$.

Therefore, by **limit comparison test**, convergence of $\sum d_n$ implies $\sum \frac{1}{a_n}$ converges.

(b) Let $(b_n)$ be the sequence given by $b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Does the series $\sum \frac{1}{b_n}$ converge or diverge? Justify your answer. [1]

**Soln.:**

Note that for each $n \in \mathbb{N}$, $b_n \leq n$, and hence $0 < \frac{1}{n} \leq \frac{1}{b_n}$.

We know that the harmonic series $\sum \frac{1}{n}$ diverges, hence by **comparison test for series**, $\sum \frac{1}{b_n}$ diverges.

(c) Prove the Sandwich Theorem for series: Suppose the terms of the three series $\sum a_n$, $\sum b_n$ and $\sum c_n$ satisfy for every natural number $n$, $a_n \leq b_n \leq c_n$, .

{Warning: Do not assume $0 \leq a_n \leq b_n \leq c_n$.}

i. If $\sum a_n = \sum c_n$, then using sequences of partial sums, show that $\sum b_n$ coverges. [1]

**Soln.:**

Let $s_n = a_1 + \cdots + a_n$, $t_n = b_1 + \cdots + b_n$, and $u_n = c_1 + \ldots + c_n$, be the partial sums of the corresponding series. Since $\sum a_n$ and $\sum c_n$ converge, the sequences $(s_n)$ and $(u_n)$ converge. For every natural number $n$, we have $a_n \leq b_n \leq c_n$, and therefore, $s_n \leq t_n \leq u_n$. As $\sum a_n = \sum c_n = \alpha$ (say), the sequences $(s_n)$ and $(u_n)$ converge to $\alpha$.

Therefore by **sandwich theorem for sequences**, the sequence $(t_n)$ is convergent and converges to $\alpha$. Therefore, $\sum b_n$ converges to $\alpha$.

ii. If $\sum a_n \neq \sum c_n$, then show that $\sum b_n$ coverges. [2]

**Soln.:**

Let $s_n = a_1 + \cdots + a_n$, $t_n = b_1 + \cdots + b_n$, and $u_n = c_1 + \ldots + c_n$, be the partial sums of the corresponding series. Since $\sum a_n$ and $\sum c_n$ converge, the sequences $(s_n)$ and $(u_n)$ are Cauchy.

Now, $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ implies $\displaystyle\sum_{i=n+1}^{m} a_i \leq \sum_{i=n+1}^{m} b_i \leq \sum_{i=n+1}^{m} c_i$ for $m > n$.

Therefore, $s_m - s_n \leq t_m - t_n \leq u_m - u_n$. $\quad \ldots(\star)$

Let $\epsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that $|s_m - s_n| < \epsilon$ for all $m, n \geq N_1$ and $|u_m - u_n| < \epsilon$ for all $m, n \geq N_2$.

Let $N = \max\{N_1, N_2\}$. For all $m > n \geq N$, we have

$$-\epsilon < s_m - s_n \text{ and } u_m - u_n < \epsilon,$$

Using $(\star)$, we now have

$$-\epsilon < s_m - s_n \leq t_m - t_n \leq u_m - u_n < \epsilon.$$

Thus, the sequence $(t_n)$ is Cauchy and hence convergent. Therefore, the series $\sum b_n$ is convergent.

3. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$. For every $n \in \mathbb{N}$, suppose there is a real number $\delta > 0$ such that for all $x \in \mathbb{R}$ satisfying $|x - c| < \delta$, we have $|f(x) - f(c)| < \frac{1}{n}$. Then show that $f$ is continuous at $c$. [1]

**Soln.:**

Let $\epsilon > 0$. Then by Archimedean Property, there exists $N \in \mathbb{N}$ such that $\frac{1}{\epsilon} < N$, i.e. $\frac{1}{N} < \epsilon$.

Now there exists $\delta > 0$ such that for $|x - c| < \delta$, we have $|f(x) - f(c)| < \frac{1}{N} < \epsilon$.

Thus, $f$ is continuous at $c$.

(b) For each natural number $n$, let $a_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n}$. Does the sequence $(a_n)$ converge or diverge? Justify your answer. [2]

**Soln.:**

For $n \in \mathbb{N}$, consider

$$\begin{aligned} a_{n+1} - a_n &= \left(\frac{1}{n+2} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2(n+1)}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{1}{(2n+1)(2n+2)} > 0. \end{aligned}$$

Thus the sequence $(a_n)$ is a monotonically increasing.
Further, for each natual number $k$, we have $\frac{1}{n+k} < \frac{1}{n}$. Therefore,

$$\begin{aligned} a_n &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \\ &< \frac{1}{n} + \cdots + \frac{1}{n} \qquad n \text{ times} \\ &= \frac{n}{n} = 1 \end{aligned}$$

Thus $(a_n)$ is bounded above.
Therefore, by **Monotone Convergence Theorem**, it is convergent.

(c) Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be given by $f(x) = \dfrac{1}{x^2} + \dfrac{1}{x}$. Find a real number $\delta$ such that for every $x$ satisfying $|x - 1.5| < \delta$, we should have $|f(x) - f(1.5)| < 0.5$. [2]

**Soln.: Attempt 1:** Take $\delta = 2$. Then $a = 3$ satisfies $|a - 1.5| < 2$. But

$$\begin{aligned} |f(a) - f(1.5)| &= \left|\frac{1}{3^2} + \frac{1}{3} - \frac{10}{9}\right| \\ &= \left|\frac{-2}{3}\right| = \frac{2}{3} > 0.5. \end{aligned}$$

Thus, $\delta = 2$ does not work.

**Attempt 2:** Check if $\delta = 1$ and $\delta = 0.5$ work!

**A short proof:**

We have to find a $\delta > 0$ such that for all $x \neq 0$ satisfying , $1.5 - \delta < x < 1.5 + \delta$, we should have $f(1.5) - 0.5 < f(x) < f(1.5) + 0.5$. Now $f(1.5) = \frac{10}{9}$.
So, for all $x \neq 0$ satisfying, $1.5 - \delta < x < 1.5 + \delta$, we should have $\frac{10}{9} - 0.5 < f(x) < \frac{10}{9} + 0.5$.
That means we should have $\frac{11}{18} < f(x) < \frac{29}{18}$.
Take $\delta = \frac{1}{4}$. (I found this by trial and error)
Then for $x$ satisfying, $\frac{5}{4} = 1.5 - \frac{1}{4} < x < 1.5 + \frac{1}{4} = \frac{7}{4}$, we have

(1) $$\frac{4}{7} < \frac{1}{x} < \frac{4}{5}$$

(2) $$\frac{16}{49} < \frac{1}{x^2} < \frac{4}{5}$$

(1) and (2) implies, $\frac{11}{18} < \frac{44}{49} < f(x) = \frac{1}{x^2} + \frac{1}{x} < \frac{36}{25} < \frac{29}{18}$.
So this $\delta = \frac{1}{4}$ will do the job.

**Detailed Solution:**

Let $c = 1.5 = \frac{3}{2}$. Then $f(c) = \left(\frac{1}{(\frac{3}{2})}\right)^2 + \frac{1}{(\frac{3}{2})} = \frac{10}{9}$.

Now $f(x) = \left(\frac{1}{x^2} + \frac{1}{x}\right) = \left(\frac{1}{x} + \frac{1}{2}\right)^2 - \frac{1}{4}$

$$
\begin{aligned}
& |f(x) - f(1.5)| < 0.5 \\
\Leftrightarrow \quad & -0.5 < f(x) - f(1.5) < 0.5 \\
\Leftrightarrow \quad & f(1.5) - 0.5 < f(x) < f(1.5) + 0.5 \\
\Leftrightarrow \quad & \frac{10}{9} - \frac{1}{2} < \frac{1}{x^2} + \frac{1}{x} < \frac{10}{9} + \frac{1}{2} \\
\Leftrightarrow \quad & \frac{11}{18} < \left(\frac{1}{x} + \frac{1}{2}\right)^2 - \frac{1}{4} < \frac{29}{18} \\
\Leftrightarrow \quad & \frac{11}{18} + \frac{1}{4} < \left(\frac{1}{x} + \frac{1}{2}\right)^2 < \frac{29}{18} + \frac{1}{4} \\
\Leftrightarrow \quad & \frac{31}{36} < \left(\frac{1}{x} + \frac{1}{2}\right)^2 < \frac{67}{36} \\
\Leftrightarrow \quad & \sqrt{\frac{31}{36}} < \left(\frac{1}{x} + \frac{1}{2}\right) \quad \text{and} \quad \left(\frac{1}{x} + \frac{1}{2}\right) < \sqrt{\frac{67}{36}} \\
\Leftrightarrow \quad & \sqrt{\frac{31}{36}} - \frac{1}{2} < \frac{1}{x} \quad \text{and} \quad \frac{1}{x} < \sqrt{\frac{67}{36}} - \frac{1}{2} \\
\Leftrightarrow \quad & \frac{\sqrt{31} - 3}{6} < \frac{1}{x} \quad \text{and} \quad \frac{1}{x} < \frac{\sqrt{67} - 3}{6} \\
\Leftrightarrow \quad & x < \frac{6}{\sqrt{31} - 3} \quad \text{and} \quad \frac{6}{\sqrt{67} - 3} < x \\
\Leftrightarrow \quad & x \in \left(\frac{6}{\sqrt{67} - 3}, \frac{6}{\sqrt{31} - 3}\right)
\end{aligned}
$$

Note that all the statements above are equivalent and hence if $x \in \left(\frac{6}{\sqrt{67}-3}, \frac{6}{\sqrt{31}-3}\right)$, $|f(x) - f(1.5)| < 0.5$. Thus if we take $\delta$ as

$$
\begin{aligned}
\delta &= \min\left\{\frac{3}{2} - \frac{6}{\sqrt{67} - 3}, \frac{6}{\sqrt{31} - 3} - \frac{3}{2}\right\} \\
&= \frac{3}{2} - \frac{6}{\sqrt{67} - 3}
\end{aligned}
$$

Thus any $x$ satisfying $|x - 1.5| < \delta$ will satisfy $|f(x) - f(1.5)| < 0.5$.
Are there any other correct answers to $\delta$?
Take $\delta_1 = 0.1, \delta_2 = 0.23, \delta_3 = 0.34289$. If $|x-1.5| < \delta_1$ or $|x-1.5| < \delta_2$ or $|x-1.5| < \delta_3$, then $|x - 1.5| < \delta$ and hence $|f(x) - f(1.5)| < 0.5$.
From the calculations above, if your $\delta > \frac{3}{2} - \frac{6}{\sqrt{67}-3}$, your answer is incorrect. Hence every correct answer to $\delta$ is a positive real number less than $\frac{3}{2} - \frac{6}{\sqrt{67}-3}$.