1. Intermediate Value Property for a subset of $\mathbb{R}$. An *interval* is a non–empty subset of $\mathbb{R}$ satisfying the INTERmediate VALue Property. Examples for intervals.

2. Exhibition theorem for intervals: Every interval is of exactly one of the following forms: (1) $\{a\} = [a, a]$ (2) $[a, b]$ (3) $(a, b]$ (4) $[a, b)$ (5) (a, b) (6) $(-\infty, b]$ (7) $[a, \infty)$ (8) $(-\infty, b)$ (9) (a, ∞) (10) $(-\infty, \infty) = \mathbb{R}$, for some reals $a < b$. Proof: Use sup/inf.

3. IVP and Continuity: Recall that a subset $A \subset \mathbb{R}$ is a *domain* if for every $a \in A$, there exists an interval $I \neq [a, a]$ of $\mathbb{R}$ such that $a \in I \subset A$. We expand our study of continuous functions to those defined on domains.

   (a) A continuous function preserves IVP: Let $f : A \to B$ be a continuous function for some subsets $A, B \subset \mathbb{R}$. If A has IVP, then $f(A)$ has IVP. Note that this does not imply B has IVP.

   (b) A continuous function takes an interval to an interval.

   (c) (Weak Preservation of Intervals Theorem) If $f : I \to \mathbb{R}$ is a continuous function defined on an interval I, then Range(f)=$f(I)$ is an interval.

   (d) (Strong Preservation of Intervals Theorem) If $f : A \to \mathbb{R}$ is a continuous function on a domain A, then for any interval $I \subset A$, $f(I)$ is an interval.

   (e) (Bolzano's Intermediate Value Theorem) Let $f : A \to \mathbb{R}$ be a continuous function on a domain A such that the interval $[a, b] \subset A$. Case (I) For any real v satisfying $f(a) < v < f(b)$ there exists a real $i \in [a, b]$ such that $f(i) = v$. Case (II) Analogous.

   (f) (Trap Root Theorem) Let $f : A \to \mathbb{R}$ be a continuous function on a domain A such that the interval $[a, b] \subset A$. If $f(a)f(b) < 0$, there exists a $z \in (a, b)$ such that $f(z) = 0$.

4. Can you prove the equivalence of the latter four statements?

5. Trap Root Algorithm: For a function satisfying the above hypotheses, find a zero with an error of less than a real $\epsilon > 0$. The procedure is

   INITIALIZE left$_1$=a, right$_1$=b, mid$_1$=$\frac{1}{2}$(left$_1$+right$_1$), error$_1$= $\frac{1}{2}$(right$_1$−left$_1$).
   DO

   IF f(mid$_n$)=0, THEN a root has been located.

   SET left$_{n+1}$ = $\frac{1}{2}$(left$_n$+mid$_n$), right$_{n+1}$ = $\frac{1}{2}$(mid$_n$+right$_n$)

   SET mid$_{n+1}$ =mid$_n$, error$_{n+1}$= $\frac{1}{2}$(right$_{n+1}$−left$_{n+1}$)

   (On a machine, exit LOOP.)

   IF (f(mid$_n$)>0), THEN

   IF f(left$_n$ > 0), THEN SET left$_{n+1}$=mid$_n$, right$_{n+1}$=right$_n$.

   IF f(left$_n$ < 0), THEN SET left$_{n+1}$=left$_n$, right$_{n+1}$=mid$_n$.

   SET mid$_{n+1}$=$\frac{1}{2}$(left$_{n+1}$+right$_{n+1}$), error$_{n+1}$= $\frac{1}{2}$(right$_{n+1}$−left$_{n+1}$)

   IF (f(mid$_n$)<0), THEN

   IF f(left$_n$ > 0), THEN SET left$_{n+1}$=left$_n$, right$_{n+1}$=mid$_n$.

   IF f(left$_n$ < 0), THEN SET left$_{n+1}$=mid$_n$, right$_{n+1}$=right$_n$.

   SET mid$_{n+1}$=$\frac{1}{2}$(left$_{n+1}$+right$_{n+1}$), error$_{n+1}$= $\frac{1}{2}$(right$_{n+1}$−left$_{n+1}$)

   WHILE error$_{n+1} \geq \epsilon$.

   When the procedure exits on iteration N, take mid$_{N+1}$ as an approximation for the root. Either f(mid$_{N+1}$)=0 in which case we have the root. Or $|\text{mid}_{N+1} - z| <$error$_{N+1} < \epsilon$, in which case we have the root to desired accuracy. (Note the $N + 1$, not N)

6. Proof of Trap Root Theorem: Use the procedure to define sequences left$_n$, mid$_n$, right$_n$ and error$_n$ inductively. Verify that left$_n$ and right$_n$ are monotonic and bounded and hence convergent to say L and R. Since

   $|\text{right}_n - \text{left}_n| = 2\cdot\text{error}_n = \frac{2(b-a)}{2^n}$, by squeeze theorem $L = R = z$, say. Further, since left$_n$ < mid$_n$ < right$_n$, by squeeze theorem again mid$_n \to z$. We claim that $f(z) = 0$. For a proof, note that in the special case if f(mid$_k$)=0 for some natural k, then, by definition, mid_n=mid_k for all $n \geq k$. Hence mid_n is an eventually constant sequence converging to z with $f(z) = 0$. If f(mid_n)$\neq 0$, for every natural n, then f(left_n)f(right_n) < 0 for every n and continuity implies $f(z)f(z) \leq 0$ and being real $f(z) = 0$.

7. Now, write a program in a suitable language, compile and run for a few functions.

1. For real numbers $a < b$, the intervals of the form $[a, b]$ are called *compact intervals*.

2. Especial Property of compact intervals: If α_n is a sequence in any compact interval $[a, b]$, then there exists a subsequence $\alpha_{n_k} \to \alpha \in [a, b]$. Proof: Bolzano–Weierstrass and Squeeze.

3. Boundedness Theorem: A continuous function on a compact interval is bounded. If $f : [a, b] \to \mathbb{R}$ is unbounded, then there exists a sequence $\alpha_n \in [a, b]$ such that $|f(\alpha_n)|$ is an unbounded monotonic sequence. By Especial Property, a subsequence $\alpha_{n_k} \to \alpha \in [a, b]$ which implies that $|f(\alpha_{n_k})|$ converges to $f(\alpha)$ by continuity. This contradicts unboundedness of $|f(\alpha_{n_k})|$.

4. Lemma: If $S \subset \mathbb{R}$ is bounded, there exists sequences (y_n) and (z_n) s.t. $y_n,\ z_n \in S$ for every natural n and $y_n \to \sup S$ and $z_n \to \inf S$.

5. Min–Max Theorem/Attainment Theorem: A continuous function $f : [a, b] \to \mathbb{R}$ on a compact interval $[a, b]$ for real $a \leq b$ attains its supremum and infimum.
   By the lemma, there is a sequence (α_n) such that $\alpha_n \in [a, b]$ and $f(\alpha_n) \to \sup f$. By the Especial Property, there exists a subsequence $\alpha_{n_k} \to \alpha \in [a, b]$. Being a subsequence, $f(\alpha_{n_k}) \to \sup f$ and by continuity, $f(\alpha_{n_k}) \to f(\alpha) = \sup f$. Done. Likewise with inf.

6. Compactness Theorem: if the domain of a continuous function is a compact interval, then its range is a compact interval. Proof: Now, the range is a subset of $[\inf f, \sup f]$, by definition of inf/sup. By Attainment Theorem, the end points are in the range. But since the range is an interval, the range is exactly $[\inf f, \sup f]$.

7. Converse/contrapositive of above property/theorems?

8. Remark: Especial Property fails for every other type of interval. These three theorems fail for continuous functions on other types of intervals and also for discontinuous functions on compact intervals. Provide explicit examples.

   The tutorial problem 7 of Tutorial Sheet #4 is useful in understanding the above statement.

9. Fake proof of boundedness theorem: Start with continuity at $x = a$ and get $\delta_1 > 0$ such that $|f(x) - f(a)| < 1$ for all $|x - a| < \delta_1$. Next continuity at $x = a + 0.99\delta_1$ yields a $\delta_2 > 0$ such that $|f(x) - f(a + 0.99\delta_1)| < 1$ for all $|x - (a + 0.99\delta_1| < \delta_2$. These two together give $|f(x) - f(a)| < 2$ for all $|x - a| < 0.99\delta_1 + 0.99\delta_2$. Continue this argument and reach b. Done. Where is the flaw?

10. Fake proof leads to the definition of uniform continuity as a convenient hypothesis to obtain boundedness. A function $f : A \to \mathbb{R}$ on a domain A is *uniformly continuous* if for every real $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x, y \in A$ satisfying $|x - y| < \delta$, it should be true that $|f(x) - f(y)| < \epsilon$. A uniformly continuous function is continuous. Prove that a uniformly continuous function on a bounded interval is bounded. Further a continuous function on a compact interval is uniformly continuous.

1. Let I and J be intervals. For functions $f : I \to J$ for $A \subset \mathbb{R}$ definitions of *increasing, decreasing, monotonic, strictly increasing, strictly decreasing, strictly monotonic.*

2. Examples for: monotonic $\not\Rightarrow$ injective, strictly monotonic $\not\Rightarrow$ surjective (continuous), continuous $\not\Rightarrow$ monotonic (injective, surjective), injective $\not\Rightarrow$ monotonic (continuous), surjective $\not\Rightarrow$ monotonic (continuous).

3. Failure to provide examples contradicting: strictly monotonic $\Rightarrow$ injective, continuous and injective $\Rightarrow$ strictly monotonic, monotonic and surjective $\Rightarrow$ continuous, monotonic and bijective $\Rightarrow$ inverse is strictly monotonic.

4. If you are giddy, here is the general principle: Suppose $A_1, A_2, A_3, \ldots A_n$ are adjectives which can be applied to a collection of nouns (objects) in N. It is only natural to ask the questions:

   (a) First Order: For any $x \in N$, if x is $A_1 \stackrel{?}{\Rightarrow} x$ is A_2 and similar $n(n-1)$ questions.

   (b) Second Order: For any $x \in N$, if x is A_1 and x is $A_2 \stackrel{?}{\Rightarrow} x$ is A_3 and similar $\frac{1}{2}(n)(n-1)(n-2)$ questions.

   (c) Third Order: Formulate and count.

   (d) Fourth–$(n-1)^{\text{th}}$ Order: Formulate and count.

   Show that the total number of such questions is $n(2^{n-1}-1)$.

5. In our example, our nouns are 'functions from intervals to intervals'. Our six adjectives are 'monotonic', 'strictly monotonic', 'injective','surjective','bijective', 'continuous'. How many questions of 1–5 orders can I ask you? How many can you answer?

6. Injective & continuous implies strictly monotonic.
   Let $f : I \to J$ be continuous and injective from an interval I to an interval J. Then f is strictly monotonic.
   Proof: Assume f is not strictly monotonic. Then f is not strictly increasing or strictly decreasing (and hence there exists $\xi_1 < \xi_2$ and $\xi_3 < \xi_4$ such that $f(\xi_1) \leq f(\xi_2)$ *and* $f(\xi_3) \geq f(\xi_4)$). This implies (fill the gap!) we can find three points in the domain $x_1 < x_2 < x_3$ such that either (i) $f(x_1) \leq f(x_2) \geq f(x_3)$ or (ii) $f(x_1) \geq f(x_2) \leq f(x_3)$. Injectivity of f allows us to infer that either (i) $f(x_1) < f(x_2) > f(x_3)$ or (ii) $f(x_1) > f(x_2) < f(x_3)$ is true. Case (i) has three sub-cases: (i.a) $f(x_1) < f(x_3)$, (i.b) $f(x_1) = f(x_3)$ and (i.c) $f(x_1) > f(x_3)$. Sub-case (i.b) violates injectivity. For other two sub-cases, apply intermediate value property to get contradictions to injectivity. Case (ii) is similar.

7. Invertible & monotonic implies monotonically invertible.
   $f : A \to B$ be an increasing bijective function from a set A to a set B. Then the inverse of f is strictly increasing. Like wise for a decreasing bijective function.
   Proof: Let $g : B \to A$ be the uniquely defined inverse of f. Suppose $b_1 < b_2$ are any two elements in B. Since g is injective, $g(b_1) = g(b_2)$ is not allowed. Suppose $g(b_1) > g(b_2)$. f being increasing and injective, we get $f(g(b_1)) > f(g(b_2))$, i.e., $b_1 > b_2$, a contradiction. Conclude that $g(b_1) < g(b_2)$.

8. Lemma on monotonic and discontinuous

   Let $f : (a,b) \to B$ be monotonically increasing. For each $c \in (a,b)$, define

   $$L_c := \{f(x) | x < c\}, \quad l_c := \sup L_c \quad \text{and} \quad R_c := \{f(x) | x > c\}, \quad r_c := \inf R_c.$$

   Then, f is continuous at $c \in (a,b)$ if and only if $l_c = f(c) = r_c$.

   Remark: Existence of l_c and r_c are guaranteed by the observations that L_c is non–empty and bounded above by $f(\frac{c+b}{2})$ while R_c is non–empty and bounded below by $f(\frac{a+c}{2})$. Indeed, $f(c)$ is an upper bound for L_c and $f(c)$ is a lower bound for R_c which implies $l_c \leq f(c) \leq r_c$.

   (continuity at c implies equality) Proof: If $l_c \leq f(c) < r_c$, find a sequence (x_n) such that $x_n > c$, $x_n \to c$, and $f(x_n) \to r_c \neq f(c)$, contradicting continuity of f at c. Similarly, if $l_c < f(c) \leq r_c$, find a sequence (x_n) such that $x_n < c$, $x_n \to c$, and $f(x_n) \to l_c \neq f(c)$, contradicting continuity of f at c. From the previous two contradictions, conclude $l_c = f(c) = r_c$.

   (equality of l_c and r_c implies continuity) Given $l_c = f(c) = r_c$ and an $\epsilon > 0$. Since $f(c)$ is the supremum of L_c, $f(c) - \epsilon < f(c)$ is not an upper bound. Therefore there exists an $x_1 < c$ such that $f(x_1) > f(c) - \epsilon$. Likewise, there exists an $x_2 > c$ such that $f(x_2) < f(c) + \epsilon$. By monotonicity, for all x satisfying $x_1 \le x \le x_2$, we have $f(c) - \epsilon < f(x_1) \le f(x) \le f(x_2) < f(c) + \epsilon$. Thus $\delta = \min(c - x_1, x_2 - c)$ works to prove continuity at c for given ϵ.

Question: How would you handle domain of f of the type $(a, b], [a, b)$ or $[a, b]$?

9. Monotonic & surjective implies continuous.
   Let $f : A \to B$ be a monotonic and surjective function from a domain A to an interval B. Then f is continuous.
   Proof: Assume f is discontinuous at $c \in A$, is increasing and apply previous Lemma. Either $l_c < f(c)$ or $f(c) < r_c$. In the former case, using monotonicity of f check that $(l_c, f(c))$ is not in the range of f, B. Now, notice that for $a < c$, $f(a) < l_c$ is in B, $f(c)$ is in B – contradicts the fact that B is an interval, satisfying intermediate value property. The other case of $f(c) < r_c$ is similar, with $(f(c), r_c)$ not in the range of f.

   The case of f being a decreasing function is analogous. And handling discontinuities at end points of A, requires care.

10. Invertible & continuous implies continuously invertible.

    Inverse Function Theorem (in the world of continuous functions): Let I and J be intervals and $f : I \to J$ a bijective continuous function. Then its inverse $g : J \to I$ is continuous.
    Proof: Relying on the previous three results, f is injective and continuous implies f is strictly monotonic, f is monotonic and bijective implies g is monotonic, g is monotonic and surjective implies g is continuous. Done.

11. Application: Existence and continuity of n–th root functions.