1. If we want to apply Cauchy's definition of convergence, we need to know the limiting point in advance. However, in many cases we may guess a sequence to be convergent without knowing the limiting point. {Example: A sequence (x_n) defined by $x_1 = 1; \quad x_2 = 2; \quad x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$: See text for full discussion} To handle such cases, is it possible to give a criterion which, if true, would enable us to conclude convergence of the sequence? Note that this criterion should be in terms of the given sequence only.

2. Such a criterion was given by Cauchy. It is based on differences between terms of the sequence.

3. Before we see that condition, let us prove: If a sequence $b_n \to b$ then, the sequences $b_{n+1} - b_n,\ b_{n+2} - b_n,\ b_{n+3} - b_n, \ldots \to 0$. Thus, if $b_n \to b$, then given any natural p the sequence $b_{n+p} - b_n \to 0$. Thus if a sequence is convergent, its p–th difference sequence should converge to 0 for every natural p. Let us call this *poochy* for short. Thus being poochy is necessary for being convergent.

4. Warning: *poochy* is a non-standard term. Please do not use it after the end of this semester!

5. Does poochy imply convergent? The example of $\sum \frac{1}{n}$ illustrates that this is not true. We need something stronger than poochy.

6. Cauchy found such a condition by making the difference sequences converge to zero uniformly in p. See tutorial sheet #2 for more inspiration.

7. A sequence (c_k) is *cauchy* if for every (real) $\epsilon > 0$, there exists a natural number N such that for all $m, n \geq N$, $|c_n - c_m| < \epsilon$.

8. Proposition: Convergent implies cauchy.

   Proof: Let c be the limit of a cauchy sequence (c_k). Given any real $\epsilon > 0$, there exists a natural number K such that for all $k \geq K$, $|c_k - c| < \frac{\epsilon}{2}$. Now, using triangle inequality: For all $m, n \geq K$, we have $|c_m - c_n| = |c_m - c - (c_n - c)| \leq |c_m - c| + |(c_n - c)| < \epsilon$.

9. Proposition: Cauchy implies bounded.

   Proof: Taking $\epsilon = 1$, there exists a natural number K such that for all $m, n \geq K$, $|c_m - c_n| < 1$.

   Thus for all $n \geq K$, $-1 + c_K < c_n < 1 + c_K$. Set $L = \min(c_1, c_2, \ldots, c_{K-1}, -1 + c_K)$ and $U = \max(c_1, c_2, \ldots, c_{K-1}, 1 + c_K)$ and all terms of the sequence are between L and U.

10. Proposition: Cauchy implies convergent.

    Proof: Given a cauchy sequence (c_n), define the sequence of infimums $\alpha_n := \inf\{c_k | k \geq n\}$ Prove the following inequalities
    $$\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_n \leq \cdots.$$
    By the Monotone Convergence Theorem, $\alpha_n \to \alpha^*$ for some real α^*.

    Given a positive real ϵ, by definition of cauchy sequence, there exists a natural L such that for all natural $m, n \geq L$, we have $|c_m - c_n| < \frac{1}{3} \cdot \epsilon$. . . . (1)
    Since $\alpha_n \to \alpha^*$, there exists a natural M' such that $\alpha^* - \frac{1}{3} \cdot \epsilon < \alpha_{M'} \leq \alpha^*$. However, the sequence of infimums is increasing and we can find a natural $M \geq \max(M', L)$ such that $\alpha^* - \frac{1}{3} \cdot \epsilon < \alpha_M \leq \alpha^*$, i.e., $|\alpha_M - \alpha^*| < \frac{1}{3} \cdot \epsilon$.. . . (2)
    Since $\alpha_M + \frac{1}{3} \cdot \epsilon > \alpha_M$, it is not the infimum of the set $\{c_k | k \geq M\}$. Hence, there exists a natural $N \geq M$ such that $\alpha_M \leq c_N < \alpha_M + \frac{1}{3} \cdot \epsilon$, i.e., $|c_N - \alpha_M| < \frac{1}{3} \cdot \epsilon$. . . .(3)
    Now, combine (1), (2) and (3) in the following.
    Given any natural $m \geq N$, $|c_m - \alpha^*| \leq |c_m - c_N| + |c_N - \alpha_M| + |\alpha_M - \alpha^*| < \epsilon$.

11. For extra-credit:

    Define the sequence of supremums (β_n) by $\beta_n := \sup\{c_k | k \geq n\}$. Prove the following inequalities
    $$\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_3 \leq \beta_2 \leq \beta_1.$$
    By the Monotone Convergence Theorem, $\beta_n \to \beta^*$ for some real β^*. One can prove that $\alpha^* = \beta^*$.

12. Remark: The α^*, β^* in the above proof are respectively termed the $\liminf$ and the $\limsup$ of the given sequence. You can try to prove that a given sequence (c_n) is convergent if and only if these two $\liminf c_n$ and the $\limsup c_n$ exist and are equal. This may be viewed as an alternative to cauchy's formulation of convergence criterion.