Section 7.1 First-Order Predicate Calculus

Predicate calculus studies the internal structure of sentences where subjects are applied to predicates existentially or universally.

A **predicate** describes a property of the subject or subjects of a sentence. If \( p \) is a predicate that describes a property of \( x \), then we write \( p(x) \).

**Example.** If \( p \) is the “is prime” predicate and \( x \) is prime, then we write \( p(x) \) for “\( x \) is prime”.

**Example.** If \( q \) is the “is a parent of” predicate, and \( x \) is a parent of \( y \), then we write \( q(x, y) \) for “\( x \) is a parent of \( y \).”

**Existential Quantifier.** The phrase, “there exists an \( x \) such that \( p(x) \)” is denoted by \( \exists x \ p(x) \). The symbol \( \exists \) is an existential quantifier and it indicates disjunction.

**Example.** If \( x \in \{1, 2, 3\} \), then \( \exists x \ p(x) = p(1) \lor p(2) \lor p(3) \).

**Quiz (1 minute).** \( p(y, a) \lor p(y, b) \lor p(y, c) = ? \)
**Answer:** \( \exists x \ p(y, x) \), where \( x \in \{a, b, c\} \).

**Universal Quantifier.** The phrase, “for every \( x \), \( p(x) \)” is denoted by \( \forall x \ p(x) \). The symbol \( \forall \) is a universal quantifier and it indicates conjunction.

**Example.** If \( x \in \{1, 2, 3\} \), then \( \forall x \ p(x) = p(1) \land p(2) \land p(3) \).

**Quiz (1 minute).** \( p(a, x) \land p(b, x) \land p(c, x) = ? \)
**Answer:** \( \forall y \ p(y, x) \) where \( y \in \{a, b, c\} \).

**Quiz (2 minutes).** If \( x, y \in \{0, 1\} \), then \( \exists x \ \forall y \ p(x, y, z) = ? \)
**Answer 1:** \( \forall y \ p(0, y, z) \lor \forall y \ p(1, y, z) = (p(0, 0, z) \land p(0, 1, z)) \lor (p(1, 0, z) \land p(1, 1, z)) \).
**Answer 2:** \( \exists x \ (p(x, 0, z) \land p(x, 1, z)) = (p(0, 0, z) \land p(0, 1, z)) \lor (p(1, 0, z) \land p(1, 1, z)) \).
Syntax of wffs in first-order predicate calculus

Terms are nonlogical things: constants $a, b, c, \ldots$, variables $x, y, z, \ldots$, and function symbols applied to terms $f(a), g(x, f(y)), h(c, z), \ldots$.

Atoms are predicate symbols applied to terms $p(x), q(a, f(x)), \ldots$.

Wffs are either atoms or if $U$ and $V$ are wffs then the following expressions are also wffs:

- $\neg U$, $U \land V$, $U \lor V$, $U \rightarrow V$, $\exists x\ U$, $\forall x\ U$, and $(U)$.

Hierarchy in the absence of parentheses

- $\neg$, $\exists x$, $\forall x$ (highest, group rightmost operator with smallest wff to its right)
- $\land$
- $\lor$
- $\rightarrow$ (lowest, and it is left associative)

Example. $\forall x\ \neg \exists y\ p(x, y) \rightarrow \forall x\ q(x) = (\forall x\ (\neg (\exists y\ p(x, y))) \rightarrow (\forall x\ q(x))$.

Scope, Bound, and Free

The scope of $\exists x$ in $\exists x\ W$ is $W$. The scope of $\forall x$ in $\forall x\ W$ is $W$. An occurrence of $x$ is bound if it occurs in either $\exists x$ or $\forall x$ or in their scope. Otherwise the occurrence of $x$ is free.

Example. $\exists x\ p(x) \rightarrow \forall y\ q(x, y)$

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Interpretations

An *interpretation* for a first-order wff consists of a nonempty set $D$, called the *domain*, together with an assignment of the symbols of the wff as follows:

1. Predicate letters are assigned to relations over $D$.
2. Function letters are assigned to functions over $D$.
3. Constants are assigned to elements of $D$.
4. Free occurrences of variables are assigned to elements of $D$.

*Notation:* If $x$ is free in $W$ and $d \in D$, then $W(x/d)$ denotes the wff obtained from $W$ by replacing all free occurrences of $x$ by $d$. We also write $W(d) = W(x/d)$.

*Example.* Let $W = \forall y (p(x, y) \rightarrow q(x))$. Then $W(x/d) = \forall y (p(d, y) \rightarrow q(d))$.

Truth Value of a Wff

The truth value of a wff with respect to an interpretation with domain $D$ is obtained by recursively applying the following rules:

1. An atom has the truth value of the proposition obtained from its interpretation.
2. Truth values for $\neg U, U \land V, U \lor V, U \rightarrow V$ are obtained by applying truth tables for $\neg, \land, \lor, \rightarrow$ to the truth values for $U$ and $V$.
3. $\forall x W$ is true iff $W(x/d)$ is true for all $d \in D$.
4. $\exists x W$ is true iff $W(x/d)$ is true for some $d \in D$.

If a wff is true with respect to an interpretation $I$, we say the wff *is true for $I$*.

*Example.* Let $W = p(x)$. We can define an interpretation $I$ by letting $D = \mathbb{N}$, $p(x)$ means $x$ is odd, and $x = 4$. Then $W$ is false for $I$ because $W(x/4) = p(x)(x/4) = p(4) = "4 is odd,“$ which is false. If we let $J$ be the same as $I$ except that we assign $x = 3$, then $W$ is true for $J$ because $W(x/3) = p(x)(x/3) = p(3) = "3 is odd,"$ which is true.
Example. Let $W = \forall x (p(x) \rightarrow q(x, y))$. Here are two interpretations of $W$:

1. Let $I$ be defined by $D = \{0, 1\}$, $p(0) = \text{True}$, $p(1) = \text{False}$, $q(0, 1) = \text{True}$, otherwise $q(x, y)$ is false, and $y = 1$. Then $W$ is true for $I$ because it becomes

$$\forall x (p(x) \rightarrow q(x, 1)) = (p(0) \rightarrow q(0, 1)) \land (p(1) \rightarrow q(1, 1))$$

$$= (\text{True} \rightarrow \text{True}) \land (\text{False} \rightarrow \text{False}) \equiv \text{True} \land \text{True} \equiv \text{True}.$$ 

2. If $J$ has any domain $D$, $p$ is true, $q$ is false, and $y$ any value of $D$, then $W$ is false for $J$.

Example. Let $W = \exists x (p(x) \land q(x))$. Here are two interpretations of $W$:

1. Let $I$ be defined by $D = \mathbb{N}$, $p(x)$ means $x$ is prime, and $q(x)$ means $x$ is odd. Then $W$ is true for $I$ because, for example, $p(3) \land q(3) = \text{True} \land \text{True} \equiv \text{True}$.

2. If $J$ consists of $D = \mathbb{N}$, $p(x)$ means $x$ is even, and $q(x)$ means $x$ is odd, then $W$ is false for $I$ because, for example, $p(3) \land q(3) = \text{False} \land \text{True} \equiv \text{False}$.

Example. Let $W = \forall x (g(x, c) \rightarrow \exists y (p(y) \land d(y, x)))$. Here are two interpretations of $W$:

1. Let $I$ be defined by $D = \{a\}$, $p(a) = \text{False}$, $d(a, a) = \text{True}$, $g(a, a) = \text{True}$, and $c = a$. Then $W$ is false for $I$ because $W = g(a, a) \rightarrow p(a) \land d(a, a) \equiv \text{True} \rightarrow \text{False} \land \text{True} \equiv \text{False}$.

2. Let $I$ be defined by $D = \mathbb{N}$, $g(x, c)$ means $x > c$, $p(x)$ means $x$ is prime, $d(x, y)$ means $x$ divides $y$, and $c = 1$. This gives the sentence, “every natural number greater than 1 has a prime divisor,” which is known to be true.

Example. Let $W = \forall x \forall y (\neg (p(x) \land p(y)) \rightarrow \exists z q(z, x, y))$. Let $I$ be defined by $D = \mathbb{N}$, $p(x)$ means $x = 0$ and $q(z, x, y)$ means $z = \gcd(x, y)$. Then the meaning of $W$ wrt $I$ is

“Every pair of natural numbers that are not both zero has a greatest common divisor,” which is known to be true.
Models and Countermodels
An interpretation that makes a wff true is called a model. An interpretation that makes a wff false is called a countermodel. See the previous examples.

Example. Let $W$ be the following wff.
$$\forall x \, (r(x, a) \land \neg p(x) \rightarrow \exists y \, (r(x, y) \land r(y, a) \land d(y, x))).$$

1. Any interpretation for which $r$ is always false is a model for $W$.
2. Any interpretation for which $r$ is always true, $p$ is always false, and $d$ is always false is a countermodel for $W$.

Quiz (1 minute). For the preceding example, let $I$ be the interpretation defined by $D = \mathbb{N}$, $r(x, y)$ means $x > y$, $p(x)$ means $x$ is prime, $d(y, x)$ means $y$ divides $x$, and $a = 1$. Is $I$ a model or a countermodel for $W$?

Answer: We get the statement,

“every natural number $x > 1$ that is not a prime has a divisor $y$ between 1 and $x$,”

which is known to be true. So $I$ is a model of $W$.

Validity
A wff is valid if every interpretation is a model. Otherwise the wff is invalid. A wff is unsatisfiable if every interpretation is a countermodel. Otherwise the wff is satisfiable.

Quiz (1 minute). Every wff has exactly two of these four properties. What are the possible pairs of properties that a wff can have?

Answer: \{valid, satisfiable\}, \{unsatisfiable, invalid\}, and \{satisfiable, invalid\}
Proofs of Validity or Unsatisfiability
We can’t check every interpretation (there are too many). So we need to reason informally with interpretations.

Example: \( \forall x \ (p(x) \rightarrow p(x)) \) is valid because \( p(x) \rightarrow p(x) \) is true for all interpretations.

Example: \( \forall x \ (p(x) \land \neg p(x)) \) is unsatisfiable because \( p(x) \land \neg p(x) \) is always false.

The previous two examples were quite simple. Here is a more realistic example.

Example. Prove the following wff is valid.

\[
\exists x \ (A(x) \land B(x)) \rightarrow \exists x \ A(x) \land \exists x \ B(x).
\]

Direct Proof: Let \( I \) be an interpretation with domain \( D \) for the wff and assume that the antecedent is true for \( I \). Then \( A(d) \land B(d) \) is true for \( I \) for some \( d \in D \). So both \( A(d) \) and \( B(d) \) are true for \( I \). Therefore both \( \exists x \ A(x) \) and \( \exists x \ B(x) \) are true for \( I \). So the consequent is true for \( I \). Thus \( I \) is a model for the wff. Since \( I \) was an arbitrary interpretation for the wff, every interpretation for the wff is a model. Therefore, the wff is valid. QED.

Indirect Proof: Suppose, BWOC, that the wff is invalid. Then there is a countermodel \( I \) with domain \( D \) for the wff. So the antecedent is true for \( I \) and the consequent is false for \( I \). Since the antecedent is true for \( I \), it follows that \( A(d) \land B(d) \) is true for \( I \) for some \( d \in D \).

Since the consequent is false for \( I \), either \( \exists x \ A(x) \) or \( \exists x \ B(x) \) is false for \( I \). So either \( A(x) \) is false for all \( x \in D \) or \( B(x) \) is false for all \( x \in D \). These cases contradict the fact that both \( A(d) \) and \( B(d) \) are true for \( I \) for some \( d \in D \). So the wff is valid. QED.

Quiz (1 minute). Is \( \exists x \ A(x) \land \exists x \ B(x) \rightarrow \exists x \ (A(x) \land B(x)) \) valid?

Answer: No. e.g., let \( D = \mathbb{N} \), \( A(x) \) mean \( x \) is even, and \( B(x) \) mean \( x \) is odd.
Some Valid Conditionals Whose Converses Are Invalid

(a) $\forall x A(x) \rightarrow \exists x A(x)$.

(b) $\exists x (A(x) \land B(x)) \rightarrow \exists x A(x) \land \exists x B(x)$.

(c) $\forall x A(x) \lor \forall x B(x) \rightarrow \forall x (A(x) \lor B(x))$.

(d) $\forall x (A(x) \rightarrow B(x)) \rightarrow (\forall x A(x) \rightarrow \forall x B(x))$.

(e) $\exists x \forall y P(x, y) \rightarrow \forall y \exists x P(x, y)$.

Quiz (5 minutes). Find a countermodel for each converse of the above wffs.

Closures

Let $W$ be a wff with free variables $x_1, \ldots, x_n$. Then we have the following definitions:

- $\forall x_1 \ldots \forall x_n W$ is the universal closure of $W$.
- $\exists x_1 \ldots \exists x_n W$ is the existential closure of $W$.

Sometimes a wff and one of its closures have the same properties and sometimes they don’t, as we can see in the following examples.

Example. The wff $p(x) \lor \neg p(y)$ is satisfiable and invalid.

- The universal closure $\forall x \forall y (p(x) \lor \neg p(y))$ is satisfiable and invalid.
- The existential closure $\exists x \exists y (p(x) \lor \neg p(y))$ is valid.

Example. The wff $p(x) \land \neg p(y)$ is satisfiable and invalid.

- The universal closure $\forall x \forall y (p(x) \land \neg p(y))$ is unsatisfiable.
- The existential closure $\exists x \exists y (p(x) \land \neg p(y))$ is satisfiable and invalid.
Closure Properties (proofs in text)
1. A wff is valid if and only if its universal closure is valid.
2. A wff is unsatisfiable if and only if its existential closure is unsatisfiable.

*Example.* The wff $p(x) \rightarrow \exists y \ p(y)$ is valid and its universal closure $\forall x \ (p(x) \rightarrow \exists y \ p(y))$ is also valid.

*Example.* The wff $p(x) \land \forall y \lnot p(y)$ is unsatisfiable and its existential closure $\exists x \ (p(x) \land \forall y \lnot p(y))$ is also unsatisfiable.

Decidability (Solvability)
A problem in the form of a yes/no question is *decidable* if there is an algorithm that takes as input any instance of the problem and halts with the answer. Otherwise, the problem is *undecidable*. A problem is *partially decidable* if there is an algorithm that takes as input any instance of the problem and halts if the answer is yes, but might not halt if the answer is no.

*The Validity Problem for Propositional Calculus*
The problem of determining whether a propositional wff is a tautology is *decidable*. An algorithm can build a truth table for the wff and then check it.

*The Validity Problem for First-Order Predicate Calculus*
The problem of determining whether a first-order wff is valid is *undecidable*, but it is *partially decidable*. Two partial decision procedures are *natural deduction* (due to Gentzen in 1935) and *resolution* (due to Robinson in 1965). We’ll study natural deduction in Section 7.3 and resolution in Chapter 9.