Solution to Problem 2.3

(a) Infinite System Capacity \((N=\infty)\)

We have \(\lambda_j = j^{-j}, \mu_j = j\mu\) for \(j=0,1,2,\ldots\).

The differential equations for each state may be written as

\[
\begin{align*}
\frac{dp_0(t)}{dt} &= -\alpha^{-0} p_0(t) + \mu p_1(t) \\
\frac{dp_j(t)}{dt} &= -(\alpha^{-j} + j\mu) p_j(t) + \alpha^{-(j-1)} p_{j-1}(t) + (j+1)\mu p_{j+1}(t) & j \geq 1
\end{align*}
\]

We define \(P(z,t)\) as the generating function of the state of the system at time \(t\) as given next.

\[P(z,t) = \sum_{j=0}^{\infty} p_j(t) z^j\]

Multiplying the \(j\)th equation by \(z^j\) and summing the L.H.S. and RHS for all values of \(j\), we will get

\[
\frac{\partial P(z,t)}{\partial t} = -\sum_{j=0}^{\infty} \alpha^{-j} z^j p_j(t) + \sum_{j=1}^{\infty} \alpha^{-(j-1)} z^j p_{j-1}(t) - \sum_{j=1}^{\infty} j\mu z^j p_j(t) + \sum_{j=0}^{\infty} (j+1)\mu z^j p_{j+1}(t)
\]

This may be simplified to get the final result

\[
\frac{\partial P(z,t)}{\partial t} = (z - 1) \left[ P\left(\frac{z}{\alpha},t\right) - \mu \frac{\partial P(z,t)}{\partial z} \right]
\]

This may be solved with the desired initial conditions to get the corresponding transient solution.

(b) Finite System capacity \(N\)

In this case, the differential equations for the system's state probabilities will become
\[
\frac{dp_0(t)}{dt} = -\alpha^0 p_0(t) + \mu p_1(t)
\]
\[
\frac{dp_j(t)}{dt} = -(\alpha^{-j} + j\mu) p_j(t) + \alpha^{-(j-1)} p_{j-1}(t) + (j + 1) \mu p_{j+1}(t) \quad 1 \leq j < N
\]
\[
\frac{dp_N(t)}{dt} = -N \mu p_N(t) + \alpha^{-(N-1)} p_{N-1}(t)
\]

From the above, multiplying the \(j\)th equation by \(z^j\) and summing the L.H.S. and RHS for all values of \(j=0, 1, \ldots, N\), we will get
\[
\frac{\partial P(z,t)}{\partial t} = -\sum_{j=0}^{N-1} \alpha^{-j} z^j p_j(t) + \sum_{j=1}^{N} \alpha^{-(j-1)} z^j p_{j-1}(t) - \sum_{j=1}^{N} j \alpha z^j p_j(t) + \sum_{j=0}^{N-1} (j + 1) \mu z^j p_{j+1}(t)
\]

This may be simplified to get the final result for the finite capacity (of \(N\)) case as
\[
\frac{\partial P(z,t)}{\partial t} = (z - 1) \left[ \frac{z}{\alpha} \frac{\partial P(z,t)}{\partial z} - \mu \frac{\partial P(z,t)}{\partial z} - \left( \frac{z}{\alpha} \right)^N p_N(t) \right]
\]