Solution to Problem 2.4

(a) The expected length of time customer A spends waiting for service = \( \frac{(n+1)}{m\mu} \)

(b) The expected length of time from A’s arrival to the time when the system becomes empty

\[ \frac{(n+1)}{m\mu} + \frac{1}{\mu} \sum_{i=1}^{m-1} \frac{1}{i} \]

(c) For \( k=1, \ldots, (n+1) \quad P[X=k] = 0 \)

For \( k=n+2, \)

\[ P[X=k] = P\{X_A < \text{residual service time of each } X_i, i=1, \ldots, (m-1) \text{ in queue} \} \]

\[ = \int_{0}^{\infty} \mu e^{-\mu \tau} [P[X > \tau]]^{m-1} d\tau = \int_{0}^{\infty} \mu e^{-\mu \tau} e^{-(m-1)\mu \tau} d\tau = \frac{1}{m} \]

For \( k=n+3, \)

\[ P[X=k] = P\{\text{one residual service time is less than } X_A \text{ while the other } (m-2) \text{ are greater than } X_A \} \]

\[ = \int_{0}^{\infty} \mu e^{-\mu \tau} (m-1)(1-e^{-\mu \tau})e^{-(m-2)\mu \tau} d\tau = \frac{1}{m} \]

In general, for \( k=n+2+i, i=0, \ldots, (m-1) \), using \( x=e^{\mu \tau} \)

\[ P[X=k] = \left( \begin{array}{c} m-1 \\ i \end{array} \right) \int_{0}^{1} (1-x)^i x^{m-1-i} \, dx \]

\[ = \left( \begin{array}{c} m-1 \\ i \end{array} \right) \sum_{j=0}^{i} (-1)^j \binom{i}{j} x^{m-1-i+j} \, dx = \left( \begin{array}{c} m-1 \\ i \end{array} \right) \sum_{j=0}^{i} \binom{i}{j} (-1)^j \frac{1}{m-i+j} = \frac{1}{m} \]

For proving the above, use the result that for \( i=0, \ldots, (m-1) \)

\[ I(m-1,i) = \int_{0}^{1} (1-x)^i x^{m-1-i} \, dx = I(m-1,i-1) \quad \text{and that } I(m-1,0) = I(m-1,1) = 1 \]

(d) Service to the customer served before customer A and the service to customer A will be as shown in Fig. 1.1 when A finishes service before the former. Here \( \tau \) is the time interval between the start of service for these two customers in the queue. Let \( X_A \) be the duration of service for customer A and let \( X_i \) be the service duration of the other customer.
The probability $P$ that we need to find is then $P= P\{\tau +X_\Lambda < X_1\}$ as obtained next where $f_\tau(t)=(m-1)\mu e^{-(m-1)\mu t}$ and $f_{X_\Lambda}(t)=\mu e^{-\mu t}$ for $t \geq 0$. Let $Y=\tau +X_\Lambda$ and since $\tau \perp X_\Lambda$, we can write that

$$L_Y(s)=L_\tau(s)L_{X_\Lambda}(s)=\frac{\mu^2(m-1)}{(s+\mu)[s+(m-1)\mu]}$$

$$=\frac{\mu(m-1)}{(m-2)}\left[\frac{1}{s+\mu}+\frac{1}{s+(m-1)\mu}\right]$$


![Figure 1.1. Service to A finishes before the service ends for the earlier customer](image)

Therefore $f_Y(y)=\frac{\mu(m-1)}{(m-2)}[e^{-\mu y} - e^{-(m-1)\mu y}]$ for $y \geq 0$.

Using this, we can find

$$P\{Y < \tau\} = \int_0^\infty f_Y(y)dy = \frac{m-1}{m-2}\left[1 - e^{-\mu \tau}\right] - \frac{1}{m-1}\left(1-e^{-(m-1)\mu \tau}\right)$$

and therefore

$$P = \int_{\tau=0}^\infty P\{Y < \tau\}\mu e^{-\mu \tau}d\tau = \frac{m-1}{m-2}\left[1 - \frac{1}{2} - \frac{1}{m-1}\left(1-\frac{1}{m}\right)\right] = \frac{1}{2}\left(1 - \frac{1}{m}\right)$$

(c) From the definition of the Erlang-$n$ distributions as the sum of $(n+1)$ i.i.d exponentially distributed random variables, we get

$$P\{w \leq x\} = \int_0^x \frac{\mu(\mu \alpha)^n}{n!}e^{-\mu \alpha}d\alpha$$